Proceedings of the 24th Solvay Conference on Physics

Quantum Theory of Condensed Matter

EDITORS

BERTRAND HALPERIN • ALEXANDER SEVRIN

World Scientific

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EDITORS

BERTRAND HALPERIN
Harvard University, USA

ALEXANDER SEVRIN
Vrije Universiteit Brussel and International Solvay Institutes, Belgium

World Scientific

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Hotel Métropole (Brussels), 11-13 October 2008

Quantum Theory of Condensed Matter

Chair: Professor Bertrand Halperin

The 24th Solvay Conference on Physics took place in Brussels from October 11 through October 13, 2007 according to the tradition initiated by Lorentz at the 1st Solvay Conference on Physics in 1911 (*Premier Conseil de Physique Solvay*). During the conference a public event was held entitled *Images from the Quantum World*. Wolfgang Ketterle and J.C. Séamus Davis delivered public lectures and a panel of scientists – consisting of Bertrand Halperin, Carlo Beenakker, J.C. Séamus Davis, Steven Girvin, Catherine Kallin, Wolfgang Ketterle, Leo Kouwenhoven, and Frank Wilczek – answered questions from the audience.

The Solvay Conferences have always benefitted from the support and encouragement of the Royal Family. His Royal Highness Prince Philippe of Belgium attended the fourth session on October 13 and met some of the participants.

The organization of the 24th Solvay Conference has been made possible thanks to the generous support of the *Solvay Family*, the *Solvay Company*, the *Université Libre de Bruxelles*, the *Vrije Universiteit Brussel*, the *Belgian National Lottery*, the *Foundation David and Alice Van Buuren*, the *Communauté française de Belgique*, de *Actieplan Wetenschapscommunicatie* of the *Vlaamse Regering*, the *City of Brussels* and the *Hôtel Métropole*.

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Opening Session

Opening Address by Marc Henneaux

Dear Colleagues, Dear Friends,

In the name of the International Solvay Institutes, it is my great pleasure to welcome all of you to the 24th Solvay Conference on Physics. This is the first Solvay Conference on Condensed Matter to be held in the 21st century.

Since I have only a few minutes, I will be brief. I will only say some words about one of the distinguished participants of the first Solvay Conference, Kamerling Onnes. This is quite appropriate at the conference that starts today since Kamerling Onnes was the champion of low temperatures (I think some colleagues nicknamed him "Mister absolute zero"). Ultra-cold temperatures have been central in the discovery of many of the phenomena that will be discussed in the coming days so one might say that Kamerlingh Onnes is one of the grand-fathers of this year's Solvay Conference.

Kamerlingh Onnes was very close to the Solvay Institutes. He took an active part in the first Solvay conference of 1911, which is the year he discovered superconductivity. He was rapporteur and was involved in many of the discussions. Lorentz had suggested that he be invited. Kamerlingh Onnes became then member of the scientific committee for physics in charge of the Solvay conferences until his death in 1926. Our archives show that he was a very active member. He was replaced by Einstein. He participated in the first four Solvay Conferences (except the one of 1924 - even though it was devoted to electrical conductivity in metals, where he could not come because he was ill but to which he sent a report on new experiments with superconductors). He was a friend of Ernest Solvay. Some of his collaborators as well as his laboratory - which was then the best for cold temperatures - received on various occasions generous financial support both from the Solvay Institutes and directly from Ernest Solvay himself. Ernest Solvay visited his laboratory several times.

I thought it was of interest to recall Kamerlingh Onnes' figure today in connection with the Solvay Institutes. The first names that come to mind when one talks about the Solvay Conferences are probably those of Lorentz, Planck, Einstein, Marie Curie, Poincaré, and later Heisenberg, Dirac, Pauli, but not his and this should perhaps be corrected.

Before giving the floor to the next speaker of the opening session, I would like to express our deepest thanks to the Solvay Scientific Committee for Physics, represented here by its chairman David Gross as well as our colleagues Tito Arecchi, Giorgio Parisi, Pierre Ramond and Klaus von Klitzing. I would like to particularly

thank to David who has been instrumental in deciding on the theme of the conference and convincing Bert to chair it. Since 2004, David has been of tremendous help and support in putting the Solvay conferences back on tracks. The success of the first Solvay conferences was due to the vision of Lorentz, who was the first chairman of the Solvay Scientific Committee for Physics. We are very lucky that David accepted to play the same role for the Solvay Conferences at the beginning of the 21st century.

I would also like to thank the rapporteurs, the session chairs and, of course, our conference chair Bert, for all the careful work that he put into scientifically organizing this meeting. The format of the Solvay conferences - which is the format Lorentz set up in the early days - is not usual since it is mostly based on discussions. This requires a lot of preparation, a clear view on what is best to favour discussions, and also a strong convincing power. It is not easy to get the reports ahead of time but Bert succeeded in getting them. The chair's task is further complicated by the fact that this is a conference by invitation only, with a strict limit on the number of participants - and given the size of the community working in the field, this is not the best way to make friends... So, I would like to reiterate the gratitude of the Solvay Institutes to Bert for having accepted to chair the 24th Solvay Conference.

Finally, I would like to make an announcement. All discussions will be recorded and transcribed in the proceedings. We have a scientific secretariat in charge of achieving this. To facilitate their task, please give your name - at least the first time - when you intervene in the discussions.

Thank you very much for your attention.

Opening Address by David Gross

Welcome everyone to the Solvay Conference. When Marc (Henneaux) asked me to chair the first of these revived Solvay Conferences and then the physics committee, I was very honoured and delighted to take on the challenge of reviving, what we all know from our study of physics and the history of physics, played such an important role in modern physics in the last century. For many of you this is the first Solvay conference and that is partly because the tradition waned a bit in the end of the 20th century and it was really due to the marvellous efforts of Marc (Henneaux) that the Solvay Institutes that support the Solvay Conferences have been rescued from the edge of bankruptcy and oblivion and this attempt is under way to restore them to their former glory. The first of the new Solvay Conferences occurred three years ago. The title of that was *The Quantum Structure of Space and Time* and that worked very well. It gave us confidence, in fact, that we could revive this very unusual setting in which some of the best minds in physics would come together for a few days in a format that would provoke them to think deeply and discuss deeply some of the important questions in our field in a setting that is so different from the normal conferences that we are all too accustomed to. Those of you who know about the Solvay Conferences, know what special occasions they were and I do urge you to read some of the proceedings of the previous conferences, they are just marvellous and even today instructive. For me it was clear that the obvious subject for the next conference should be the structure of matter even be it condensed and I could think of no one better to help organize and mould such a conference than Bert Halperin and that has proven to be correct, as we will all see. It only took a short conversation of four hours to convince Bert to agree and he has been extraordinarily diligent putting together a program with just the right people and the right spirit. Learning from last conference we have tried to even further facilitate and promote discussion, so I urge all of you even after this grandiose historical introduction, not to be shy. We do hope that these conferences will be viewed in a hundred years as having played an important role in physics, but that's no reason for you to be shy. It is a reason not just to be flipping, but we would like to encourage you to take part in the discussions that will be the main part of the meeting. Without further ado, I turn over for brief remarks by Bert.

Opening Address by Bertrand Halperin

Thank you very much David (Gross) and Marc (Henneaux),

I am indeed honored to have been given the opportunity to lead this 24th Solvay Conference on Physics. It is true, as David suggested, that I was rather hesitant at first to take on this responsibility. The prospect of trying to choose fifty participants to represent a field as broad as condensed matter physics seemed especially daunting. In organizing the conference, I did indeed have to make some rather painful decisions, as there were many people I wished I could invite but could not because of the limited space. At the end, however, I am very pleased with the group of participants we have been able to attract, and I am excited by program we have been able to arrange. I do look forward to a most successful meeting.

Why should we have a Solvay Conference on this subject at this particular time? Let me first note that a conference on quantum theory of condensed matter fits very well within the Solvay tradition. A number of early Solvay Conferences were related to this subject. The first conference, directed by Lorentz in 1911, was devoted to the general problem, *Theory of Radiation and Quanta*. The second conference, in 1913, was specifically devoted to *The Structure of Matter*. The fourth was on *Electrical Conductivity of Metals and Connected Problems*. The sixth conference, in 1930, was *Magnetism*. In 1951 there was a conference on *Solid State* under Sir Lawrence Bragg, and then in 1954, on *Electrons in Metals*. The 1978 conference, *Order and Fluctuations in Equilibrium and Non-Equilibrium Statistical Mechanics*, while it may be not have been very quantum mechanical, was certainly closely related to many of the problems we will be discussing this week. The topics *Surface Science* in 1987, and *Quantum Optics* in 1991, certainly included a lot of quantum condensed matter. *Quantum Dynamical Systems and Irreversibility*, in 1998, was again related to many issues that we will be discussing here.

This year, we have been given the opportunity to convene a Solvay Conference fully devoted to the subject of quantum theory of condensed matter. I would argue that this is an excellent time for such a conference. Condensed matter is a very important part of physics today and will be for the foreseeable future. It is important because it has practical consequences, but also, as we know, it raises many issues in basic science. Problems concerning the collective behaviour of many-particle systems have shown themselves, repeatedly, to be very subtle, and we are faced with numerous unanswered questions, including some very fundamental ones.

What can a three day conference, with just fifty participants, accomplish towards answering these questions? We recognize that there are already hundreds of conferences on various subjects within condensed matter science each year, which try to address many of these issues. What can we achieve with one more conference?

David has already said something about the Solvay tradition and how the Solvay conferences differ from other conferences. First of all, open discussion across a broad field should be a defining characteristic of any Solvay Conference. Through these

discussions, we can truly benefit from the small, select but diverse, group of participants that we have been able to bring together.

Second, we have tried hard to define the subject of the conference in a way that would maximize the potential for stimulating interchange of ideas. In order to focus discussion in a manageable area, we decided to concentrate on *quantum* aspects of condensed matter, even though we know that the line between quantum condensed matter and classical condensed matter is actually very thin. Indeed, many of the participants have gone back and forth across this line, and some are most heavily involved in classical problems. Many subtle ideas, such as notions of glassiness, are important, in both quantum and classical contexts. However, the emphasis will be on problems where quantum mechanics plays a central role.

It was also decided that the conference should emphasize theoretical developments. We recognize, of course, that experimental investigations are absolutely essential to our field. Given the time constraints of a three-day conference however, it seemed that we might accomplish most by focusing on theoretical issues. As a consequence of this emphasis, there are perhaps 10 experimentalists, compared with 40 theorists, among the participants. We trust, however, that the experimentalists will bring their own views to the discussion, and will hopefully keep the theorists on track.

During the course of the conference, I hope that participants will try to identify the most important outstanding problems facing our field. I hope we shall also exchange views on ideas that might be transferred from one branch of condensed matter to another, including computational methods, analytical methods, and physical concepts. We should also discuss implications of future technological and experimental developments that might enable us to study new materials or new kinds of devices, or to make new types of measurements. By anticipating such developments, we may be guided in looking for directions of research where the theory of quantum condensed matter is likely to receive maximum stimulus from the experimental side.

Any conference which hopes to advance a field as broad as the quantum theory of condensed matter, by bringing together fifty participants who are world leaders in the field, for a period of just three days, is necessarily an experiment in itself. I am very optimistic, however, that through the efforts of all of the participants, the experiment will prove to be a great success.

I shall not review here the titles on the program of scientific sessions, which you have already seen. I would like, however, to remind you about the Solvay format. In each session, there will be one or two talks by rapporteurs, which will last one hour in total, whose purpose is to give us an overview of a subfield. This will be followed by an hour and a half of open discussion, which I consider the most important part of the conference. I hope all the conference participants will join in these discussions. The rapporteurs, clearly, will not be able to cover everything within a field; they will select developments they believe are most central. Their presentations should stimulate other participants to add their own views. The Session Chairs will be

responsible for guiding the discussion, and for making sure that all participants have a chance to contribute.

There will be a break in the conference program on Sunday afternoon. At that time, there will be a public event called *Images from the Quantum World* which will consist of two talks, by Seamus Davis and Wolfgang Ketterle, followed by a panel discussion in response to questions from the audience. The panel discussion has been billed on the program as a debate, so this may be rather interesting. You are all cordially invited to attend the public event.

Let me again thank all of the conference participants for accepting our invitation. I appreciate that many of you have traveled a long distance to get here. I trust that you will all enjoy the conference and that you will indeed find the conference to be highly productive.

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Special Session

On the Quantum Theory of Condensed Matter

Talk delivered by Betrand Halperin at a special session of the 24th Solvay Conference on Physics, in the presence of His Royal Highness, Prince Philippe of Belgium. The talk summarizes the purpose of the conference, beginning with an explanation of the meaning of the term "condensed matter", and the role of quantum theory in its analysis. Four topics, central to discussions at the conference, are introduced as illustrations: superconductivity, nano-scale devices, collections of ultra-cold atoms, and phases and phase transitions.

1. Introduction

Your Royal Highness, Mr. and Mrs. Solvay, Ladies and Gentlemen, Colleagues and Friends: It is my pleasure and honor to present a brief introduction to the topic of the Quantum Theory of Condensed Matter, which is the subject of the 24th Solvay Conference on Physics. My purpose is to give you some idea of what the subject is all about, and what we are trying to do at the conference.

2. What is "Condensed Matter"?

Condensed matter refers to materials or structures made up of a large number of particles that are sufficiently close together so that the interactions between them are crucial for understanding their behavior. The particles here are generally electrons and atomic nuclei. Condensed matter includes nearly all the substances that we find on earth and that are important to our every day experiences and to technology.

Among the common states of condensed matter are liquids and crystalline solids, glasses and liquid-crystals, metals and insulators, semiconductors and magnets. Condensed matter science also includes the study of surfaces, of thin films, and of interfaces between different materials. It also includes the study of more complicated structures, or devices, made from multiple materials. The interests of condensed matter scientists include the understanding of the properties of materials

and devices, including electrical, mechanical, magnetic, and thermal properties, as well as the development of techniques to synthesize new materials and to fabricate new devices.

3. Why Quantum Theory

As was discovered early in the twentieth century, the classical laws of physics break down when one is concerned with the behavior of electrons on the length scale of individual atoms or molecules. A new set of rules, the laws of quantum mechanics, was found to apply under these circumstances. The laws of quantum mechanics are equivalent to to the laws of classical physics when one is dealing with the macroscopic motion of sufficiently large objects. However, the special features of quantum mechanics are essential for understanding the forces between atoms, which are responsible for the microscopic structure of materials, and therefore for determining such macroscopic properties as the rigidity of a solid or the electrical conductivity of a metal or semiconductor. These microscopic inter-atomic forces, of quantum-mechanical origin, are, in turn, essential for understanding all of chemistry, including the chemical reactions which underlie biological and geological processes in nature, and which may be exploited by man to synthesize new materials.

Quantum mechanical behavior can also be manifest directly on length scales larger than the molecular scale under appropriate circumstances. This is particularly likely at low temperatures. When quantum mechanics is important at larger length scales, the results are often surprising and fascinating. A large part of the discussion at this conference is focused on such situations.

4. The Solvay Tradition

Our conference on Quantum Theory of Condensed Matter is very much in the tradition of past Solvay Conferences. Among them were a large number whose topics relate closely to the current subject. For example:

The first Solvay Physics Conference, organized by H. A. Lorentz in 1911, was entitled "Theory of Radiation and Quanta." This conference was devoted to discussions of the great puzzles of the time, whose eventual solution would lead to the development of the quantum mechanics we know today. Many of the great physicists of the time, who would contribute to the development of quantum mechanics, were present at the 1911 conference, and at several of the succeeding conferences as well. The interchange of ideas that occurred at these conferences clearly played a role in the development of the quantum theory.

The second Solvay Physics Conference, in 1913, on the the "Structure of Matter", was another landmark in the developments leading up to quantum theory and its application to the properties of matter. The fourth conference, in 1924, was devoted to the "Electrical Conductivity of Metals", a subject which is at the heart of condensed matter science even today. Both of these conferences were chaired by Lorentz.

The sixth Solvay Conference in Physics, chaired by Paul Langevin in 1930, was devoted to "Magnetism". By this time, the basic theory of quantum mechanics was fully developed. The subject of magnetism was an area of condensed matter physics where quantum mechanics had had an enormous impact, as it was essential to any understanding of the phenomenon. Yet there were outstanding puzzles, and the subject of magnetism still remains very active today.

When the Solvay Conferences resumed after the end of World War II, quantum mechanics and condensed matter subjects again played a role. The ninth Solvay Physics Conference, organized by William Lawrence Bragg in 1951 was entitled "The Solid State". The tenth conference, in 1954, also organized by Bragg, was concerned with "Electrons in Metals".

In more recent years, there have been several other Solvay Physics Conferences with subjects related to the quantum theory of condensed matter. For example, the seventeenth, organized in 1978 by Léon Van Hove, was concerned with order and fluctuations, in equilibrium and non-equilibrium, in statistical mechanics. The nineteenth conference, organized in 1987, was concerned with surface science.

The twentieth conference, organized by P. Mandel in 1991, was concerned with "Quantum Optics", the study of quantum-mechanical interactions between light and matter. Many of the experimental tools used to study condensed matter systems depend on the interactions between light and matter. Moreover, in recent years, laser beams have been used to exert forces on matter and even to create new forms of condensed matter systems. An exotic type of condensed matter system made possible by interaction of light and matter, a collection of ultra-cold atoms in an optical trap, is one of the topics discussed at the current Solvay Conference.[a]

5. Why This Subject, Now?

The quantum theory of condensed matter is important both because of the fundamental questions it raises, and because it may have significant implications for practical applications. Improved understanding of condensed matter systems in the quantum realm may be a key to solving a number of technological problems of vital importance to society. (Some current applications will be mentioned below.) Although there has been enormous progress in our understanding of condensed matter since the development of quantum mechanics in the 1920s, there are many issues that remain poorly understood, and many new questions that are not yet answered.

Much of the improvement in our understanding of condensed matter systems has come from theoretical developments, including the introduction of new mathematical techniques and computerized computational methods. Even more important has been the contribution of experiments. Laboratory scientists have synthesized new substances and have learned how to make materials, even old familiar materials, with much higher purity and uniformity than ever before. They have learned

[a]See the proceedings of Session 5.

to fabricate devices of exceedingly small size, and they have developed extremely powerful tools to make measurements that could never be done before. The new tools have given us enormous insight into the wide ranging properties of condensed matter systems, but they have also uncovered a multiplicity of surprising results, whose fuller understanding is the object of much of our current research.

The current Solvay Conference on the Quantum Theory of Condensed Matter is an occasion for a select group of scientists, including many of the leaders in the field, to get together and exchange ideas about the most outstanding questions confronting the subject.

6. Focus on Collective Effects

Condensed matter systems are constructed, on the microscopic level, out of simple particles, electrons and atomic nuclei, which are essentially point-like. The microscopic quantum-mechanical equations which govern their motion, have been known since the 1920s. But the consequences of these equations, particularly when one has to consider large numbers of interacting particles, can be very subtle, and are only partially understood.

Interactions between many particles can give rise to "collective effects", where the behavior of the resulting material is very different from what one would have expected if one imagined that the particles moved in a more-or-less independent way. Many of the most current interesting problems in condensed matter physics, and the topics of much of the discussion at the current Solvay Conference, are directly related to such collective effects in systems of many particles.

In the following sections, I will give four examples of major subjects that will be explored at the conference: superconducting materials, nano-scale devices, collections of ultra-cold atoms, and phases and phase transitions.

7. Superconductivity

Superconductivity was discovered, experimentally, in 1911 by H. Kamerlingh-Onnes, who used his newly invented technique for producing liquid helium to study the properties of materials at very low temperatures, never before possible in the laboratory. (Kammerlingh-Onnes was, incidentally, a very important figure in the early history of the Solvay Conferences.) Superconductivity was first observed in mercury; then, soon after, in tin and lead. In each case, it was found that below a certain temperature specific to the material, known as the *critical temperature* or T_C, the material will enter a "superconducting state", where it can conduct an electric current without any loss of energy. This is in contrast to the normal metallic state, where there is a non-zero electrical resistance, and a current flow is always accompanied by dissipation, in which a portion of the electrical energy is converted to heat.

Superconductivity is a collective effect, which was finally understood in 1957, with the theory of Bardeen, Cooper, and Schrieffer. In essence, the electrons in a

superconductor are bound into pairs, and these pairs "condense" into a new state, where the very large number of pairs are described by a single collective quantum-mechanical "wave function". In normal materials, where electrons are not paired, resistance occurs because individual electrons are scattered, one at a time, in collisions with impurities or vibrational distortions of the crystal. In a superconductor, individual electrons cannot scatter because they are bound into pairs; the pairs cannot scatter one by one, because they are all forced to belong to single wave function. To change this wave function, which describes simultaneously a very large number of electron-pairs, would take a large amount of energy, which is not available from scattering events unless the electrical current itself is very large. However, if the current is made larger than a critical value, the *critical current*, superconductivity will be lost, and the material will behave like an ordinary metal.

Between 1911 and 1985, many new superconductors were discovered, but the highest critical temperatures were below 30 K (that is 30 degrees above absolute zero.) Temperatures in this range can only be reached, in practice, using liquid helium as a coolant, and elaborate thermal insulation, which is too expensive for most applications. In 1986, a new class of materials was discovered, the "high T_c cuprates", for which critical temperatures are commonly in the range of 95 K, almost four times as high as any previously known T_c. (Cuprates are materials where the superconductivity is generated in crystal planes containing copper and oxygen atoms.) Temperatures in the useful range for cuprate superconductors can be reached using liquid nitrogen as a coolant, which is very much cheaper than liquid helium. Thus, one can imagine a wide range of new applications employing high-temperature superconductors.

Conventional superconductors are already being used, despite the expense of helium cooling, for certain essential applications, where no alternative is available. These applications include particularly the use of superconducting wire to produce high field magnets, for scientific research and for medical applications such as medical resonance imaging (MRI) machines used in hospitals around the world. High temperature superconductors could be useful for building powerful electric motors and for lossless electric power transmission, as well as for a variety of specialized electronic applications.

So far, commercial applications of high-temperature superconductors have been slow in coming, not because of the expense of cooling the materials with liquid nitrogen, but because the materials themselves have proven difficult to work with. Only recently have we learned how to produce large quantities of superconductor with large critical current, at reasonable cost. However, commercial pilot projects using high-temperature superconductors for electric power transmission are currently under way. A particularly attractive potential application is to replace existing copper power cables in the underground conduits that service major urban areas with superconducting cable, which could carry significantly more electric current in the same limited space. I think we can expect many more applications of high temperature superconductors in the future.

The highest known T_c, which was found in one of the cuprate superconductors under high pressure, is about 150 K, about half of room temperature. If a material can be found that exhibits superconductivity at room temperature, and has other important properties like a high critical current and reasonable manufacturing cost, the technological implications could be staggering. Even superconductors with transition temperatures similar to the cuprates, but better material properties, could be very important. However, we do not know what really limits the maximum achievable T_c, and in fact, we still do not understand very well the microscopic mechanism for superconductivity in the cuprates.

The cuprates actually belong to a much wider class of materials, where the repulsions between electrons are very strong. In these materials, an electron can move from one atom to another only by pushing another electron out of its way. Even the non-superconducting states of these materials are very peculiar, and are not well understood. For example, there is much debate about the proper way to describe the cuprates in the "normal state", above their critical temperatures. Strongly interacting materials exhibit a number of other peculiar properties, including magnetic properties, that might be eventually important for technology. Very recently, a new class of superconductors has been discovered, based on layers containing iron and arsenic atoms, rather than copper and oxygen, where strong electron-electron repulsion is also believed to be important. Although the transition temperatures in these new materials are not yet as high as in the cuprates, there is exciting potential here for further development.

Several sessions of the 24th Solvay Conference on Physics are concerned with the theory of high T_c cuprates and related materials with strong electron-electron repulsion.[b]

8. Nano-Scale Devices

Much of the world's economy today is based on technology made possible by a dramatic reduction of size in the electronic devices at the heart of digital computation, such as transistors and magnetic memories. This reduction in size has been the key to higher speeds, improved performance, and reduction in cost. If current trends continue, devices will soon become so small that current engineering principles can no longer work. It is crucial that we understand the limitations of current electronic devices and discover principles for new types of devices at the smallest possible length scales.

To give you an idea of the ongoing reduction in the size of electronic devices, consider the recent evolution of transistor gate lengths in commercial state-of-the-art silicon computer chips. According to the 2007 International Technology Roadmap for Semiconductors,[1] the gate length in 2007 was about 25 nm. [A nanometer (nm) is one billionth of a meter.] In 1995, the gate length was 400 nm. The gate length

[b]See, particularly, Sessions 2 and 3.

projected for 2015 is 10 nm, which is only 40 times the diameter of a silicon atom in the underlying chip!

In very small devices, quantum mechanics may become important in subtle new ways. A very small device may behave quite differently than a larger version of the same device. Several sessions of this conference are devoted to the study of electronic devices that are extremely small – in one, two, or all three of their dimensions.[c]

9. Collections of Ultra-Cold Atoms

Although quantum mechanics is often crucial for understanding the behavior of electrons, it is not usually necessary for describing the free motion of atoms. Atoms are generally more than 10,000 times heavier than electrons, and quantum mechanics is usually much less important for heavy objects. Recently, however, it has become possible to trap collections of atoms and cool them to incredibly low temperatures – of the order of one billionth of a degree above absolute zero! Under these conditions quantum mechanics is essential for the description of system. Also, rather remarkably, the interactions between atoms may play an important role in their motion even though the atoms in these traps are so dilute that they are, on average, very far away from each other.

Despite the huge difference between the masses of electrons and atoms, systems of ultra-cold atoms can display quantum phenomena quite similar to those observed in electron systems. Phenomena which have already been observed include collective behavior very analogous to the superconductivity of electrons in superconducting metals.

Even more interesting, it is possible by a trapping atoms in an optical standing wave, produced with intersecting laser beams, to create atomic systems whose mathematical description is identical to that of simplified models that have been proposed to describe electrons in cuprate superconductors and other strongly-interacting electron materials. These models, although much simplified from a full description of the actual cuprate materials, are still so complicated that their properties can only be deduced using approximate theories that are somewhat controversial. If models can be built successfully using ultracold atoms, we should be able control parameters of the model in a way that is not possible for the cuprates, and we may employ a number of experimental tools unavailable in the electronic case, to obtain a better understanding of the properties of the model systems. This will enable us to determine which, if any of the proposed models are adequate for describing the cuprates and will help us to determine which of the proposed approximate theories are reliable for predicting the properties of the models.

Systems of ultra-cold atoms, and their connections to phenomena in electron systems, are one of the major subjects of our Solvay conference.[d]

[c]See, particularly, Sessions 1 and 4.
[d]See, particularly, Session 5.

10. Phases and Phase Transitions

Materials can exist in different collective states, or *phases* such as solids, liquids and gases, with very different physical properties. Under certain circumstances, a small change in conditions may cause a substance to convert from one phase to another, with a drastic alteration of its properties. For example, a small increase in temperature can cause a solid to melt into a liquid, completely losing its property of rigidity. Such a change of state is called a *phase transition*. Other phase transitions include the change of state that occurs when a superconductor is heated above its critical temperature, losing its property of superconductivity, or when a magnet such as iron is heated above its critical temperature, where it loses its magnetic properties.

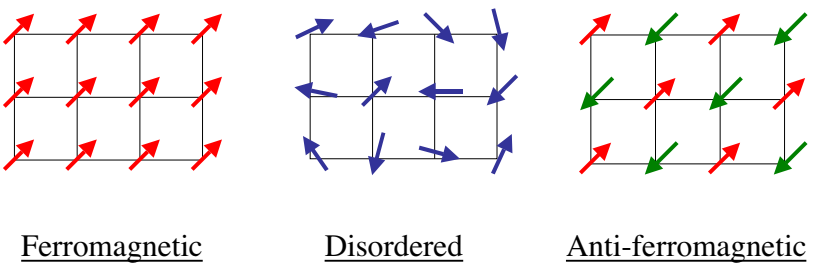

Fig. 1. Schematic representation of spin order in various magnetic phases.

Many phase transitions represent a change in the ordering of atom positions or of electronic degrees of freedom. For example, various magnetic phases are characterized by different arrangements of the orientations of the microscopic magnetic moments arising from the electron's quantum mechanical spin. In Figure 1, we show schematically the way these moments are in arranged in several magnetic phases. Here we have drawn the magnetic atoms in two dimensions, as arranged on the sites of a square lattice. Each atom has a magnetic moment, arising from the spin of its electrons, whose orientation is indicated by the arrows in the figure. In the left hand panel, we illustrate a *ferromagnetic* phase, where the spins all tend to line up in the same direction, due to an attractive interaction between them. This is the magnetic phase of iron at room temperature. The central panel illustrates a *disordered* phase (or *paramagnetic* phase), where the magnetic spins point in random directions, and there is no correlation between the directions of spins separated by more than a few lattice distances. This would describe the electron spins of iron in its non-magnetic phase, above the critical temperature for magnetism. In the right-hand panel we show an *anti-ferromagnetic* phase, in which the magnetic moments of atoms point in opposite directions on adjacent lattice sites. Here, moving along a row of atoms, one would encounter alternating spin directions: up, down, up, down,..... , continuing in

a predictable alternating pattern over large distances.

Many other types of ordering and phase transitions have been found to occur, particularly at low temperatures, in systems where quantum mechanics is important. Some of these phases are quite exotic, and are very difficult to describe in ordinary language. Fundamental questions discussed at the Solvay Conference on the Quantum Theory of Condensed Matter include: What new types of phases are yet to be discovered, and what would be the nature of the transitions between such phases?[e]

11. Current Technologies based on Quantum Properties of Condensed Matter Systems

As was mentioned above, many technological devices in current use are based on quantum properties of condensed matter systems. We have already discussed the applications of superconductivity, and the diminishing sizes of integrated circuits and transistors. Quantum condensed-matter science was also essential for the development of solid state lasers, used in medicine, in optical communication, and in devices to read and right CDs and DVDs. An understanding of the quantum properties of condensed matter systems was crucial for the design of modern magnetic memories, and for the sensors used read out information from hard disk magnetic storage devices. Quantum properties of solid surfaces are key to the functioning of catalysts in many industrial chemical reactions. Although most catalysts today have been created by empirical methods, one could imagine a situation where better catalysts could be designed based on improved fundamental understanding of surfaces and the catalytic processes. Quantum properties of materials have many possible applications in the design of devices for storage or conversion of energy, such as improved storage batteries or photoelectric devices for solar energy conversion.

Knowledge gained from the study of condensed matter systems where quantum mechanics is important will surely have other applications in the future that we cannot foresee today. The primary purpose of this Solvay meeting is to advance our fundamental knowledge of condensed matter systems, but we must also recognize the likelihood that this knowledge will yield major benefits for technology and society in the longer run.

References

1. *The International Technology Roadmap for Semiconductors: 2007 Edition.* Executive Summary, p. 62. Posted by ITRS at http://www.itrs.net/reports.html .

[e]See Sessions 2, 3, 4 and 5.

Session 1

Mesoscopic and Disordered Systems

Chair: *Daniel Loss*, University of Basel, Switzerland
Rapporteur: *Boris Altshuler*, Columbia University, USA
Scientific secretaries: *Liza Huijse* (Universiteit van Amsterdam) and *Sasha Zozulya* (Universiteit van Amsterdam)

Rapporteur talk: Mesoscopic and disordered systems

Unfortunately the write up of the rapporteur Boris Altshuler is not available.

Discussion

D. Loss Thank you very much Boris for your talk and I am sure there are now a number of questions and comments coming up.

X.-G. Wen I have a question, so in this true insulator for interacting particles do you have a transport of energy? That is if a local particle excitation interacts with the next particle excitation the energy can probably be transported, then that may be even like a phonon mode. It is kind of strange, I mean also it has a phonon, even without phonon. So, I have a little trouble here.

B. Altshuler The way we look at that is the following: you know that in the usual case people already agreed that there is no coexistence between extended and localized states, so either your energy belongs to an extended or a localized band but the probability to find extended states in a localized band is equal to zero. And when you speak about many body state you have to include all the excitations. As soon as some subsystem will get delocalized it will carry delocalization of the rest of the system, unless you

have some particular conservation law that does not allow this. So what I think is, that there is no transport of energy.

B. Halperin Could you say a little bit more about what the experimental difference would be between a bad metal and a good metal. You said that you cannot tell whether it is ergodic or not by doing a simple experiment, so how can we distinguish them?

B. Altshuler From the traditional point of view it is more difficult to distinguish a bad metal from the insulator, because a bad metal is actually very bad, it has very large resistance, much bigger than quantum resistance. But I think from the point of view of physics, the difference is of course much deeper, because a bad metal is extremely non-ergodic and transport there happens not uniformly, but in a kind of special way and a very non-uniform way. So this is something that, in principle is observable, how to do that, I do not know.

B. Halperin But if the distinction is whether it has high or low conductance compared to the quantum resistance, I am not sure that going to high temperature is going to make a difference.

B. Altshuler No, no, you are right. But still at least in the experimental situations I know, when you go to high enough temperature, the conductance never is much smaller than the quantum, it is always about the same or bigger. But what I want to tell is that, for instance, noise properties would be very different from a good metal and all things like the avalanches can happen and so on.

A. Stern When you talk about the classical limit, it is the limit where you do not care whether you have bosons or fermion in your system, right? So bosons in that limit would form a bad metal as well?

B. Altshuler Yes, I think a boson can form a bad metal, although of course for them it is more difficult than for fermions because they like to get together with many of them, but if they are repulsive this is not a problem. The problem with the classical situation as compared to the quantum case, is that you can transfer energy by arbitrarily small portions and because of that you can benefit from resonances which are very high in frequency. So if two oscillators have incommensurate frequencies the ratio of these frequencies is close to some rational number with a very big denominator, still it will work, because you can transfer a very small amount of energy. Now when you are in a quantum system, where the particles are discrete and the energy transfer is only in discrete portions, very high resonances do not help and this is to our mind the main difference between classical and quantum transport in this situation.

A. Stern So the statistics of the particle does not matter?

B. Altshuler No, I mean, it matters in the details. But the fact that there is no transition, so only a bad metal, no insulator in classical physics, is based on the fact that you can transfer energy with arbitrarily small portion, not

on the statistics.

E. Shimshoni I wanted to ask about the connection to the experiments on the disordered thin films. What do you expect, would be the ingredient in the realistic solid state system that could give you an observation of this insulator?

B. Altshuler There are very many experiments already existing which may be interpreted in different ways. What I think is legitimate to ask is what would be a proof of this statement. And the first thing, I think experimentalists should do, and I hope they will do that, is to come up with a certain way to measure electron temperature directly. So if there would be direct experimental evidence that electrons are overheated in the region where resistance is very high, then it would be a proof that transport is not by phonons and then it is very likely that we will find a way of deciding whether it is a true insulator or a bad metal.

E. Shimshoni But I guess what I am asking is what would give you the suppression of the phonons, how can you get rid of them?

B. Altshuler This is a good question, but my point is that it is not about suppressing phonons, suppose some phonons are there, the question is whether transport is just traditional phonon hopping or not. And the answer is that it is very likely that it is not, because I cannot imagine how you can explain these experiments, in spite of the fact that the resistance is very high, by phonon assisted hopping.

M. Fisher I just want to get a clarification on Xiao-Gong (Wen)'s question, which is, are you saying the thermal conductivity is also zero in your state in which ... (*interrupted by Altshuler*)

B. Altshuler Yes, moreover I think that if there would be finite thermal conductivity it would mean that there are some delocalized excitations that are there and then these excitations at finite temperature can surface above.

M. Fisher The other question I had is, what is the effect of an unbounded spectrum, which I guess in any physical system you do not have a tight binding model, you always have an unbounded spectrum. In particular in 3D versus 2D versus 1D ... (*interrupted by Altshuler*)

B. Altshuler The only thing I can tell is that if there are extended states somewhere high, but the distance is volted, we are at 10 mK, I mean we are not in O.J. Simpson's trial where ten to the minus thousand is different from zero.

M. Fisher Let me ask then about an unbounded spectrum in two dimensions, where the localization length is finite. ... (*interrupted by Altshuler*)

B. Altshuler It still diverges. In any finite system size there is an energy at which the localization length exceeds the system size, so there are some states above this energy, which are not different from extended states in 3D.

J. Chalker So you said that the bad metal and the good metal should be separated by a crossover and I can see that thinking about the conductivity that seems

reasonable, but you also said that in a bad metal a state should be non-ergodic and in a good metal they should be ergodic and that sounds like a sharp distinction. So I wonder how you put those things together.

B. Altshuler No, it is not sharp because even if we are allowing ourselves to take the limit of N to infinity you still can ask yourself what is the definition of support. So there is an average value of the density $|\psi|^2$ divided by the number of sites. So will you call the site belonging to that if it is 1 percent of that or 10^{-4} or 10^{-5}? I mean if you change the definition you can get from one to another, so this definition is not mathematically rigorous. I think the reason why we believe that it is a crossover and not a transition is, first of all, not a proof, but our good guess, but what we have in mind is that the difference between these two is basically in the value of the fluctuation compared to the average one and what happens is that in good metals fluctuations are very small and in bad metal they are very large and the average can be not representative. But it is not a proof, it is just some kind of feeling.

J. Chalker The picture that I had is that we have at least some simple states which are extended but not ergodic from what we know about single particle problems and the mobility edge and we can characterize them in terms of multi-fractal behaviour and so on and then I think I would imagine a set of multi-fractal exponents which could evolve as I varied some parameter there would still be a distinction between a truly ergodic and something sub-ergodic.

B. Altshuler No, there is a distinction, but I do not think you can identify a boundary to that, that this energy plus epsilon is clearly and this energy minus epsilon is clearly non-ergodic. If you can, let us discuss it.

D. Loss Now the part two of the mesoscopic session is to start and we will try to cover also other topics in mesoscopic physics. Mesoscopics is defined in a very broad sense. I even looked it up this morning on Wikipedia. It is defined nowadays almost identical to nanoscale systems and if you take this definition you can imagine that list of topics here which I collected from few of you will not be very short. Just to give you a first impression of topics of interest in that field, in particular, are hyperfine interaction, nuclear spins have received lots of interest, quantum dynamics of these systems is of great interest both theoretically and experimentally, hyperfine induced orderings in nuclear spin systems is something of great interest. Then quantum computing in solid state systems. I am not reading every item just trying to give you a first impression. Then new materials, new systems, as you can read here, nanowires, carbon-based materials, many new systems, also hybrid systems are looked at very intensively in that field and that even can lead into some questions which would be interesting to discuss here, that basically quantum engineering can be one direction of the future. Then what I can call low dimensional systems. There are a

number of very interesting open problems even in the most simplest case of a ballistic transport through a quantum point contact. 0.7 anomaly is a key word here. Then, Fermi edge singularities; electron, energy and phase relaxation due to magnetic impurities; non-standard Fermi-liquid behaviour in low dimensions; effects of spin-orbit interaction. The key word here is spintronics and spin-orbit interaction, Rashba-Dresselhaus type of spin-orbit, spin-electric effects, spin-Hall effect, quantum spin-Hall effect, topological insulators and so forth. So this list here is certainly not exhaustive and although it fills one slide I have made room for other suggestions. Please, I hope you will be able to fill in after the discussion more topics. But now basically let me go back to the first topic up here and ask people in the audience to make comments and possibly also give some presentations to this. I think, because I have prepared a little bit ahead and asked people actually, I think it would be a good start here to begin with graphene and particular Anderson localization considered in graphene because this would nicely connect to the previous talk. So let me ask now Carlo (Beenakker) to present his view.

C. Beenakker – prepared comment

D. Loss Thank you Carlo (Beenakker), the session is now open for questions.

E. Shimshoni What is the plot that you made of the beta-function, the guess so-to-speak, what is based on?

C. Beenakker It is based on? It is based on, let us see. This branch, this point, this limit here is a weak anti-localization, there we can do perturbation theory. Up here you can do perturbation theory in weak disorder. This is somehow the ballistic limit. And so the guess is this central bump. Because there is another guess which says that it will actually go down like this guy and then match up by going back up. So then it will be different. But it is a guess which is supported by the existing computer simulations which are not on very large systems so you could question them. I am convinced that this is true, by the way. I am convinced that this question mark is an exclamation mark, that it is really like that.

D. Loss Boris (Altshuler), do you have a comment?

B. Altshuler Carlo (Beenakker) I just want to ask. When you draw something like that it is only for short-range potential or you can allow long-range random potential as well. And, if yes, is it important or not?

C. Beenakker This applies to potentials which are smooth on the scale of lattice constants of carbon such that inter-valley scattering can be neglected. So you need smooth potentials. Once it is smooth, the smoother it is the better it is.

B. Altshuler That cannot be too smooth...

C. Beenakker No, it cannot be too smooth. It can be too abrupt, it cannot be too smooth.

B. Altshuler But if you just break your graphene sheet into two and put it into

different pockets there will be no transport. If potential is too long-range..

C. Beenakker The size of the system... I am talking about random potential which is on average is zero. So if your potential is incredibly smooth but will have some large value somewhere you will just have to make your samples bigger and bigger and bigger. If you make it incredibly smooth you will have a certain correlation length of your potential, right? You need your sample size to be much larger than the correlation length. But it is OK, I am talking about scaling, how it will scale as I scale my system to infinity.

D. Loss Good, Patrick (Lee) you have a question.

P. Lee I think we should emphasize that this curve does not apply to graphene because this is for single Dirac fermion. And as far as realization of this, I think Shinsei Ryu has a suggestion that you can have it on a surface of a 3D topological insulator.

C. Beenakker Yes, absolutely. I even had a word here, you may not have noticed it, this was a result of a brain storm session yesterday where we tried to come up with a new word for the topological insulator because we are talking about metals, right? So, Kramers metal? So it applies to graphene to the extent that the system is still small compared to the inter-valley scattering length, which could well be, potentials are rather smooth in graphene, it applies to the surface of a 3D topological insulator to a much larger accuracy, absolutely.

D. Loss OK, please.

N. Nagaosa As long as you are working on a lattice model, I think in the limit of a very strong disorder compared with the transport integral the system is always in the localized regime. Am I right?

C. Beenakker Well, no. Not if this is the Hamiltonian. If this is the Hamiltonian you are studying, this guy here, $v\vec{p}\cdot\vec{\sigma}+V$, if that is... You may say that it is not the right model. Excuse me? (*interrupted*)

N. Nagaosa I mean the lattice model.

C. Beenakker That is precisely the problem. You want to have a lattice, you want to have a finite difference representation of this Hamiltonian, which does not run into the difficulty that you mention. You cannot just discretize $\vec{p}$, say, let us take some finite difference... Then it will not work, you have to be smarter than that. In fact, that is exactly the problem. How to put this on a lattice in such a way that it does not localize no matter how large I put the inter-potential.

L. Balents I guess I have maybe a possible response to the end of the comment. Somewhat obvious way one could try to answer that question is following. Just take one of those microscopic models that are rather simple, proposed by Fu and Kane for topological insulator.Those are well defined tight-binding models but in three dimensions rather than two, which have those surface states. You could put disorder in them, there is a certain amount of additional cost to that computationally... (*interrupted*)

C. Beenakker That is one way to somehow work around the problem. I am not pursuing that, I need to go to big systems, it is logarithmic in the length, right, so I do not think that working from the 3D and then going to study just the surface is an efficient way to proceed. But it is certainly a way to end ... (*interrupted*)

L. Balents I do not know, may be the matter of choice for the lattice case...

C. Beenakker No, it is true, if you work from 3D and go to the surface that should work.

L. Balents I have a comment. Maybe additionally, I wanted to ask... This is a beta function which I also think that this notion that you cannot localize the surface is probably correct for those topological insulators, but it is certainly assuming no interactions and its stability relies on time-reversal symmetry. It seems to me that most likely as you increase disorder in a real system, which has interaction, has very likely possibility of spontaneously breaking time-reversal, at least at the surface which could ... you know. So that is something that may be worth looking into.

C. Beenakker Absolutely.

D. Loss OK, so may we take one last question or comment to graphene. Yes, please.

M. Cohen Boris (Altshuler) mentioned that for non-phonon systems he looked at electron-hole possible scattering, and I was wondering where collective excitations come in. It is particularly interesting with regard to graphene because if we do full calculations, beyond tight-binding models we find that there are strong electron-electron effects and strong coupling between the electrons and the collective excitations because in two dimensional systems, the plasmons can be at low energies. With regard to disorder, when we put in real defects like Stone-Wales defects and things of that kind, we see a lot of changes. I agree with you, maybe we should consider graphene. I was wondering, when do collective excitations come into some of these models?

B. Altshuler You see what I discussed was just kind of generic picture. And from this point of view if there is any collective excitation which is delocalized then it will eventually delocalize charge as well. At least according to this Mott scenario. What we think follows from our consideration is that there is a region where all of them are localized including collective excitations. And this is the answer. If in your calculation they are not localized it means you are above the transition and something like that. But then clearly conductivity also will be...

M. Cohen When you go from your bad metal to your good metal , what happens to the ergodic nature?

B. Altshuler No, no. In a bad metal everything is delocalized. It is not ergodic, maybe, but it is delocalized.

M. Cohen What happens in going from the insulator to the bad metal?

B. Altshuler Then indeed you delocalize, but you delocalize everything at once in a sense.

D. Loss Ok, one last comment.

L. Glazman Maybe I will paraphrase the previous question. One may think not of the excitations but of the response function. Look at the response function, OK? So you are talking about DC conductivity, but one may think about AC conductivity. And then the question would be are there any features that correspond to collective excitations? So, say, for clean systems there is a plasmon peak in $\sigma(\omega)$, what one would expect for the unusual metals?

B. Altshuler Honestly, I do not know. You have your neighbour who is a collaborator you can ask, but honestly I do not know. Because clearly it is the question about high-frequency response or even low frequency response, it is a question about resonance pairs of localized states. And when your states are localized in so complicated space, how to make any generic statement about this I do not know. So you can ask what is going on in some particular model and probably we can calculate, but not in generic sense, At least I cannot tell you.

D. Loss OK, I see no hands raising, then maybe we move on to a different topic. I suggest to have some discussion now on nuclear spins in low-dimensional systems and if I see Shankar (Das Sarma)... So, maybe you can say a few words on hyperfine interaction?

S. Das Sarma – prepared comment

D. Loss OK, thanks Shankar, comments, questions, further suggestions. Yes?

S. Sachdev If I consider the problem of just a single electron-spin and many nuclei what are the open issues there?

S. Das Sarma (*unclear*) is a single electron spin but it is many nuclei, so it is a nuclear many-body problem.

S. Sachdev So just for a single electron (*interrupted*)

S. Das Sarma Just for a single electron-spin it is a nontrivial problem.

S. Sachdev And it is coupled to all the nuclei?

S. Das Sarma That is correct. This you know in principle a localized electron-spin and the bunch of nuclear spins in the environment. So this could be the problem that my student Wayne Witzel did. He took something like 10 to 100 million spins here and basically did a cluster expansion that pretty much like classical cluster expansion, but you have to do it, so that all those nuclei making equivalent contribution. So historically Hahn... After you do spin-echo this a part of the decoherence that is still there. And Hahn realized that it is there so what he said, he said, well I am going to ignore all these complications. I am going to assume that it is Gaussian random Markovian process for the electron-spin. So it is just, Brownian motion. And then Klauder and Anderson modified, it has to be, they said, Lorentzian Markovian process. But in reality this Hamiltonian is completely non-Markovian process. So it is a complicated problem not because of electron-spin but because there are many-many nuclear spins.

D. Loss Maybe I can add some comment here, that already the hyperfine Hamil-

tonian itself summed over all the nuclear spins is a many-body Hamiltonian whose time dynamics is non-Markovian. Even without dipole-dipole part this already is a problem which cannot be solved exactly in general. There is one particular solution ...(*interrupted*)

S. Sachdev I guess my question is what is the problem?

Loss Time dynamics. So the problem is you are given the initial state of your electron spin and and you want to know how it evolves in time and how it decays. What is the time scale over which the electron spin decays.

S. Das Sarma If you put a spin in an up state and you ask for a dephasing, for example.

B. Halperin I think one should not concentrate on just a single problem of an initial state with a fixed, time-independent Hamiltonian. I mean, this is a very rich system and experimentally you can do all kinds of things with this kind of system. Typically, experiments use two electrons, rather than one. It is just slightly more complicated . But you can control enormous number of things, say by varying voltages applied to gates, and experiments are mostly non-equilibrium experiments. It really depends on what your initial state is. You are not necessarily starting with an initial state of one electron and the thermal bath. Or maybe you are, but then you have to pump the system a few times, you start polarizing the nuclei and you can polarize different nuclei, different amounts, and you can flip them over, and you can do resonance. So it is a vast problem.

Loss ...(*unclear*)I tried to answer Subir's question what is a precisely defined problem, but of course there are many more aspects ...(*interrupted*)

B. Halperin I think there is an enormous future to this and maybe it has relevance to quantum computation, but more generally there are just an enormous number of experiments that can be done, that are just beginning to be done.

S. Das Sarma I completely agree with you both, so the quantity that we are concentrated on is what will happen if you pulse it. Spin-echo is the easiest thing – you pulse it, and see how much coherence is lost. Then you can do more complicated pulsing and all of them are in some sense different problems. So I agree with you. But in the end what you are looking at is some aspect of density matrix and how it evolves in time and more complicated versions of that. And that is what I meant. It is a huge problem. It is not one single problem. And that is why I am not showing you anymore results because it depends on what you are interested in.

E. Demler Do I understand correctly that this model is in principle Bethe ansatz solvable?

S. Das Sarma No, no.

L. Glazman Sorry, I will give an answer for Daniel (Loss), who is a better specialist actually in it. So if you throw away dipole-dipole interaction, just look at what is called the central spin problem. It is one of the so-called Gaudin

magnets. There is a Bethe ansatz procedure. There are known N integrals of motion, which is in principle solves the problem. But now, I am not a specialist, but as far as I understand, there is no explicit solution for the initial condition problem. So if you tell somebody what is the initial state, and ask to tell what is the state at time t I do not think that you volunteer to answer.

E. Demler I would then formulate the question more generally. There is a wide range of problems which are Bethe ansatz solvable but we do not know how to use it to study time-dependent dynamics. And that is true not only about this central spin problem but about Lieb-Liniger, other types of problems.

S. Das Sarma Let me comment. As Leonid (Glazman) said Bethe ansatz, to the extent, it solves a class of problems when you put a dipolar coupling to zero. Once the dipolar coupling is there, which is very important because...(*interrupted*)

L. Glazman Right, let me maybe have a phrase. Without dipole-dipole interaction dynamics is believed to be non-exponential. For Subir's (Sachdev) question, if you look at a central spin decay $\langle \vec{S}(t)\vec{S}(0)\rangle$ averaged over the initial state. The decay is believed to be not exponential. There is actually a paper by Leon Balents and Doron Bergman, that have shown or at least gave a hint or made us to believe that decay is model dependent. It depends on how the wave function of the central spin, of the electron decays at large distances and it is $1/\ln t$ or $1/\ln^2 t$. But that is a result without dipole-dipole. With dipole-dipole it is a new game and to the best of my knowledge it is not integrable and perhaps there will be exponential decay. But I think it is an interesting question, because you may argue that there is spin diffusion which will bring exponential decay but spin diffusion is affected itself by the presence of central spin. It is like a strongly inhomogeneous field. So I agree that it is a very interesting open problem.

S. Das Sarma It is a very interesting problem, as Bert said, it is a very interesting set of problems not one problem depending on what question you are asking.

D. Loss OK, I should not abuse here my position. I would like to make a few more comments but may be I can do this later. Are there more comments to this nuclear spin problem? OK, please.

B. Keimer I was just going to point out that in solids there is a lot of interesting quantum phase transitions that are being studied by a lot of people. And they all take place at low temperatures in the presence of the nuclear spin bath. And I guess once you have solved this problem there is the next problem how hyperfine interactions with these nuclear spins influence quantum phase transitions in solids.

S. Das Sarma Yes, I think that is a good point.

D. Loss That is actually what we have started to study, the hyperfine interaction between many electrons and nuclear spins that induces a kind of phase transition at finite temperature already. Then the question arises how this

interferes with zero-temperature phase transition.

S. Das Sarma But very little has been done on the specific question. That is a good question.

D. Loss OK, good. So let me move on to a next topic. I think maybe a next topic of interest will be to go to some more esoteric topic like the future of quantum computing in solid state and here the main candidates in solid state are superconducting qubits, Josephson Junction qubits and also spin qubits. I think it would be nice if we could have a few words from the experts here. Maybe we start with Josephson Junction qubits and Girvin, if you would like to say a few words?

S. Girvin – prepared comment

S. Das Sarma Steve (Girvin), let me connect something you have just mentioned with what we were talking about a few minutes ago – basically the spin bath problem. So one of the issues that happens in the superconducting quantum computing is the issue of flux noise. Where does flux noise come from? You have a flux qubit and they have some intrinsic noise which some people think are coming from some intrinsic two-level systems. But one possibility is that flux noise could in principle arise from this nuclear spins flip-flopping. Because every time nuclear spin flip-flop that is temporary fluctuating magnetic field in your flux region and this is something I have planned to look at for quite a while. But I have not had a chance to get to it yet. Do you have a feel for whether this is quantitatively consistent with the amount of flux noise one has in experimental systems?

S. Girvin Well, I am not an expert on this but Lev Ioffe has looked at the model of not nuclear but electron spins with just stray trapped electrons in places near the surface, possibly getting their dynamics by being in close proximity to the metal and having RKKY-like couplings in the presence of the superconducting gap. And they claimed that, looking at John Clark's experiments and so forth, that there is a possibility that it quantitatively explains it.

S. Das Sarma But that is a bit different. Because that is basically a poisoning effect. You know the electron going in and coming out. It is not this slow fluctuations of the nuclear spins. That effect is an electron goes in, it is basically just poisoning, Andreev effect essentially, right? What Lev Ioffe is talking about...

S. Girvin Right, so people have asked themselves whether nuclear spins could be doing this and I do not think the numbers worked out, but I am not an expert on it.

D. Loss We did some estimate a few years ago and it (nuclear spins) seemed to be a weak effect not explaining actually the kind of dephasing seen in these flux qubits.

S. Das Sarma Smaller than that? OK.

D. Loss Are there more questions, comments?

A. Kitaev I want to make a comment on the last question. I am collaborating with Lev Ioffe now exactly on this problem of magnetic spin fluctuations caused by electron spins. There is actually an effect that causes very low frequency fluctuations in spin subsystems, basically if there is a dipole of nearby spins that are coupled by RKKY, this dipole can form a singlet or doublet and it is difficult to transfer energy to the bath if the bath is made of the same kind of spins. There may be some electron-spin event.

May I also say something about this Josephson Junction qubits that are protected against those fluctuations. There have been several designs and the most promising and the simplest one is due to Doucot, Vidal, Ioffe and Feigelman of Josephson junction arrays that are intrinsically stable against noises, but they are stable only to the extent that the charge noise is much smaller than one integrated over the whole array. It is still an open problem how to deal with the charge noise even in those arrays. If we can eliminate it, that would be very nice.

S. Davis Steve (Girvin), do you think there is a fundamental reason why solid state quantum computing is proving to be so difficult?

S. Girvin Well, I would say we have $1/f$ noises and flux and critical current and electric field offset that are just ubiquitious and very poorly understood, so our strategy is to design qubits that are insensitive to that. I do not know how to just get rid of it.

S. Davis It would save experimentalists a tremendous amount of trouble if there was a theorem proving that it is impossible to make a solid state quantum computer.

S. Girvin Well, there is interesting physics in the attempt.

D. Loss Actually, I would assume that such a theorem would be a challenge for you as an experimentalist to disprove it, right?

S. Das Sarma I want to just slightly rephrase what Seamus (Davis) was saying: I do not think solid state quantum computing is particularly more difficult than quantum computing per se. I do not see a quantum computer going very far with ion traps and other architectures either, so if you take the word "solid state" away from the question, I am very happy. I do not think that the problem is intrinsic to solid state. Quantum computing is turning out to be more difficult than we thought.

S. Davis So what is the answer to my question?

S. Das Sarma The answer to your question is that there are issues with quantum decoherence that are a very big challenge. I do not think there is any theorem here at all, I think it is just a question of implemental advance, which will take us somewhere.

S. Sachdev So can I follow up on that? Alexei (Kitaev) just mentioned protected Josephson Junction arrays and there is topological protection in quantum Hall systems, so is that a way of getting around the problem of decoherence? In your case you have to worry about flux noise, charge noise and so on.

A. Kitaev Theoretically all those proposals are based on assumptions and the usual assumption is that the noise rate relative to something is much smaller than one. For the Josephson Junction arrays the charge noise is particularly important, the flux noise can be tolerated. We cannot make it work unless we make the elements sufficiently good.

D. Loss Very good. Maybe we move on now to the second contribution on spin qubits. Leo (Kouwenhoven), can you say a few words on the semiconductor based spin qubits?

L. Kouwenhoven – prepared comment

D. Loss Let's discuss now a last topic before we go for a lunch break: spintronics has become a very important field in recent years in mesoscopic physics. The spin-orbit interaction has become a tool to study new effects such as spin-charge mechanisms which might lead to new devices. The keywords here are spin-currents and the spin Hall effect.

A. MacDonald Spin Hall effect or quantum Hall effect is an interesting example of localization that Boris (Altshuler) was talking about before. In the quantum Hall effect the localization problem is peculiar because you have a gap at a density that depends on magnetic field, so that leads to edge states. People have realized that in the quantum Hall effect one way of understanding it is in terms of some topology of the bands in the bulk of the system connected to these edge states. People have realized, particularly Kane and Zhang and others, that this can happen in ordinary crystals just non-interacting particles, that you can have edge states that are driven by some topological structure of the band and I think that there are fairly convincing experiments that this exists. Firstly, 2D surface states of Bismuth-Antimony for example do seem to show Dirac behaviour of the type that Carlo (Beenakker) was talking about. It should be very interesting to study localization physics and other mesoscopic physics in those systems. Secondly, there are edge states in 2D materials, like Mercury-Tellurite, studied by Molenkamp and motivated by theories of Zhang and collaborators. I think that studying the properties of edge states in those systems would be very interesting.

Maybe I can say just one thing as a comment on this. You have the 0.7 anomaly that is up there and there are many experts here on one-dimensional electron physics and if you look at 0.7 anomaly and you hear M. Pepper talk about it, for example, it sure looks like those systems are one dimensional mean-field ferromagnets in terms of their phenomenology. That seems kind of strange, one-dimensional systems are not supposed to have properties like that, of course they are not strictly one-dimensional, but I have always wondered what real experts on one dimensional physics would think about this phenomenon.

N.P. Ong I would like to pick up a bit on what Allen (MacDonald) said to publicize the work of my colleague at Princeton, (Zahid) Hasan, who did the

experiments confirming the predictions of (Charlie) Kane and (Liang) Fu on the existence of surface states in Bismuth-Antimony. According to Zahid, in angle resolved photo emission spectroscopy (ARPES), it is usually very hard to see surface states, but when he took these crystals grown in Cava's lab to the machine, he found that they just jumped out. To see the surface state alone of course is not sufficient. One has to count the number of surface states. With high resolution ARPES, Hasan found the number to be odd, in agreement with theory. This paper appeared in Nature in April (2008). Recently he has done spin resolved ARPES, and confirmed that only certain of these states have the Rashba-like term in the Hamiltonian via which the spin rotates as one goes around the surface Dirac point. These results lead us into a very interesting chapter in condensed matter physics.

M. Cohen Very quickly, just throwing in graphene again. If you consider graphene ribbons, and Allen (MacDonald) knows this, then the edge states are polarized and they are anti-ferromagnetically oriented. If you put on an electric field, you can bring either one spin or the other spin to the Fermi energy and can get 100% polarization. So these edge states in real crystals, which was mentioned, are really very very interesting.

L. Balents I felt like, as we wrap up this session, that maybe it is worthwhile to get ourselves thinking about what is new and exciting in the future and I certainly feel that these topological insulators is one aspect of that, but generally, what has driven mesoscopic systems is exquisite control and quantum engineering of structures and particularly high quality material. I like that point. If I am sort of bold and speculating, I think there is a development going on that might turn into a new chapter in mesoscopic physics, which is that people are beginning to be able to grow digital layer by layer structures of a wide variety of complex oxide materials with transition metals rather than ordinary semiconductors. One can imagine the same kind of engineering that goes on today in Gallium-Arsenide happening in a diverse variety of perovskites and even beginning to see things, like a recent experiment showing a tunable superconductor-insulator transition in a sample like this Strontium-Titanate and Lanthanum-Aluminate, I think. Maybe as theorists, since this is a theory conference, this is a direction that we should be thinking about.

D. Loss I agree and also think that quantum engineering will become increasingly more important. To some extent we can think now more concretely about the phases and quantum dynamics we would like to generate in certain nanostructures, and then to find ways to design the material with the right Hamiltonian. A good example where this goal is already pursued right now is the implementation of quantum computation in solid state systems such as semi- and superconductors, as we've just heard from Leo (Kouwenhoven) and Steve (Girvin). This has triggered quite some effort to gain control over quantum states and their time dynamics. Progress, however, seems only

possible if we understand the 'environment' of quantum states well enough, so that we can identify regimes of enhanced or even protected coherence. Given the complexity of mesoscopic effects, as shown by Boris (Altshuler) in his talk, and given the extremely high precision of control that is needed, we are here only at the beginning of this development. But to me it looks very interesting and worth pursuing.

With this outlook I'd like to thank everybody for contributing to the discussion and close the session on mesoscopics.

Session 2

Exotic Phases and Quantum Phase Transitions in Model Systems

Chair: *Antoine Georges*, Ecole Polytechnique, Palaiseau, France
Rapporteur: *Subir Sachdev*, Harvard University, USA
Scientific secretaries: *Nathan Goldman* (Université Libre de Bruxelles) and *Soo-Jong Rey* (Seoul National University)

Rapporteur talk: Exotic phases and quantum phase transitions: model systems and experiments

1. Abstract

I survey theoretical advances in our understanding of the quantum phases and phase transitions of Mott insulators, and of allied conducting systems obtained by doping charge carriers. A number of new experimental examples of Mott insulators have appeared in recent years, and I critically compare their observed properties with the theoretical expectations.

2. Introduction

The band theory of electrons predicts that any crystal with an odd number of electrons per unit cell must be a metal. However, strong electron-electron interactions can invalidate this conclusion, and such crystals can also be insulators, known as Mott insulators. I will use this term here more broadly: often the Mott insulator has a secondary instability to spin or charge ordering which increases the size of the unit cell, so that the ultimate ground state of the insulator does have an even number of electrons per unit cell (many examples of such instabilities will be dis-

cussed below). I will continue to refer to such insulators as Mott insulators because electron-electron interactions are crucial to understanding their broken symmetries and excitation spectrum. In contrast, describing the ordering by using the instabilities of a metallic state with an odd number of electrons per unit cell leads to a rather poor understanding of the insulator and of the energy scales characterizing its excitations.

A canonical model used to study Mott insulators is the single-band Hubbard model

$$H_U = \sum_{i,j} \left(-t_{ij} - \mu\delta_{ij}\right) c^\dagger_{i\alpha} c_{j\alpha} + U \sum_i n_{i\uparrow} n_{i\downarrow} \tag{1}$$

where $c_{i\alpha}$ annihilates an electron with spin $\alpha = \uparrow, \downarrow$ on the sites, i, of a regular lattice, and $n_{i\alpha} = c^\dagger_{i\alpha} c_{i\alpha}$ is the number operators for these electrons. For small U, the ground state of H_U is a metal (on most lattices) which can be described in the traditional framework of band and Fermi liquid theory. For strong repulsion between the electrons with $U/|t_{ij}| \gg 1$, and with the chemical potential μ adjusted so that there is one electron per unit cell, charge fluctuations are strongly suppressed on each site, and the ground state is a Mott insulator. The low energy excitations of the Mott insulator are described by an effective Hamiltonian which is projected onto the subspace of states with exactly one electron per site. These states are described by the spin orientation of each electron, and the effective Hamiltonian is a Heisenberg quantum spin model

$$H_J = \sum_{i<j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + \ldots \tag{2}$$

where $J_{ij} = 4t^2_{ij}/U$ is the antiferromagnetic exchange interaction, $\mathbf{S}_i$ is the spin $S = 1/2$ operator on site i, and the ellipses refer to multiple spin-exchange terms which are generated at higher orders in the expansion in t_{ij}/U. One of the purposes of this article is to survey theoretical advances in understanding the ground states of H_J on a variety of lattices in two spatial dimensions. A number of experimental realizations (some newly discovered) will also be surveyed and critically compared with theory.

More broadly, the study of models like H_U and H_J will lead us to a number of exotic phases, both insulating and conducting, which require modern concepts from gauge theory and 'topological' order for their complete characterization. Our unifying strategy here will be to access these exotic states across a quantum phase transition from a conventional state. We will begin by characterizing the 'order' in a conventional state, and then turn up the strength of quantum fluctuations leading to a quantum 'disordering' transition to an exotic state. This approach will lead to 4 broad classes of exotic states, discussed in the sections below:

(*i*) *Quantum fluctuating antiferromagnetism.* We begin with an insulator with antiferromagnetic long-range order, well described by H_J. Quantum fluctuations of the antiferromagnetism lead to states with full SU(2) spin rotation symmetry, and

an energy gap to spin excitations. In Section 3 we consider a simple, and now well-understood model: the coupled dimer antiferromagnet. In this case, well-developed methods from the theory of finite temperature phase transitions can be extended to successfully describe its ground states and quantum phase transition. In Section 4, we will introduce a recent experimental example of a triangular lattice antiferromagnet, and develop a theory for the non-magnetic insulating states in which the quantum interference effects play a more fundamental role, and new theoretical ideas are required.

(*ii*) *Neutral fermions across the Mott transition.* We begin with the Fermi liquid state of H_U, characterized by a Fermi surface (in some cases, Fermi points) of charge $\pm e$, spin $S = 1/2$ quasiparticles. Now we postulate a continuous Mott transition to an insulator in which the spin and charge of the quasiparticles separate, and a 'ghost' Fermi surface survives in the insulator, with the Fermi surface excitations carrying $S = 1/2$ spin, but no charge; these are fermionic spinons. The current status of such exotic states will be reviewed in Section 5.

(*iii*) *Breakdown of Kondo screening.* As discussed in Section 6.1, the heavy fermion state of rare-earth intermetallics is described by the Kondo-Heisenberg model describing the exchange coupling of local moments to itinerant conducting electrons. The Kondo effect tightly entangles the local spins and the itinerant electrons, leading to a 'large Fermi surface' state, which encloses a volume determined by the total electron density, including both the local and itinerant electrons. For sufficiently strong exchange between the spins, the Kondo screening can break down, and the local moments and itinerant electrons disentangle, leading to a 'fractionalized Fermi liquid'. In the simplest models, the itinerant electrons form a small, metallic Fermi surface of conventional electronic quasiparticles, while the local moments form a spinon Fermi surface.

(*iv*) *Quantum fluctuating metallic spin density waves.* We begin with a metallic Fermi liquid state, in the presence of spin density wave order. This order will generally break up the Fermi surface into electron and hole pockets. Section 6.2 will describe a quantum transition in which the spin density wave order becomes short range, but ghost Fermi pockets survive in the resulting 'algebraic charge liquid'. In the latter state, the Fermi surface excitations carry charge $\pm e$ but no spin. Such a state has been used recently to develop a theory of the enigmatic underdoped region of the cuprates.

The concluding Section 7 will survey recent experiments on Mott insulators on a number of frustrated lattices, and compare observations with numerical studies and the theoretical proposals.

3. Coupled Dimer Antiferromagnet

This model is illustrated in Fig. 1. The $S = 1/2$ spins reside on the sites of a square lattice, and have nearest neighbor exchange equal to either J or J/λ. Here $\lambda \geq 1$ is a tuning parameter which induces a quantum phase transition in the ground state

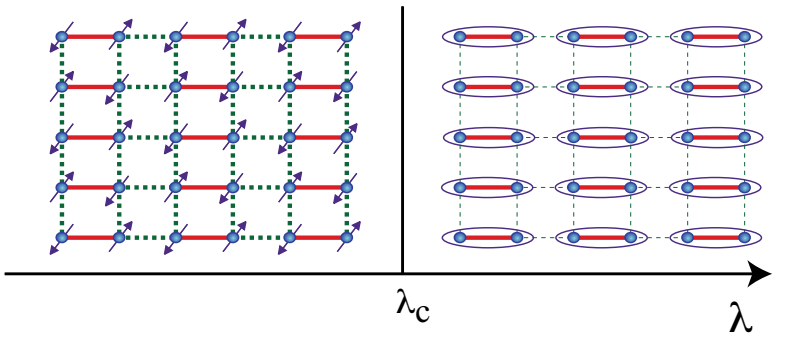

Fig. 1. The coupled dimer antiferromagnet. The full red lines represent an exchange interaction J, while the dashed green lines have exchange J/λ. The ellispes represent a singlet valence bond of spins $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$.

of this model.

At $\lambda = 1$, the model has full square lattice symmetry, and this case is known to have a Néel ground state which breaks spin rotation symmetry. This state has a checkerboard polarization of the spins, just as found in the classical ground state, and as illustrated on the left side of Fig. 1. It can be characterized by a vector order parameter $\boldsymbol{\varphi}$ which measures the staggered spin polarization

$$\boldsymbol{\varphi} = \eta_i \mathbf{S}_i \tag{3}$$

where $\eta_i = \pm 1$ on the two sublattices of the square lattice. In the Néel state we have $\langle \boldsymbol{\varphi} \rangle \neq 0$, and we expect that the low energy excitations can be described by long wavelength fluctuations of a field $\boldsymbol{\varphi}(x, \tau)$ over space, x, and imaginary time τ.

On the other hand, for $\lambda \gg 1$ it is evident from Fig. 1 that the ground state preserves all symmetries of the Hamiltonian: it has total spin $S = 0$ and can be considered to be a product of nearest neighbor singlet valence bonds on the J links. It is clear that this state cannot be smoothly connected to the Néel state, and so there must at least one quantum phase transition as a function λ.

Extensive quantum Monte Carlo simulations[1–3] on this model have shown there is a direct phase transition between these states at a critical λ_c, as in Fig. 1. These simulations have no sign problem, and so it has been possible to obtain extremely precise results. The value of λ_c is known accurately, as are the critical exponents characterizing a second-order quantum phase transition. These critical exponents are in excellent agreement with the simplest proposal for the critical field theory,[3] which can be obtained via conventional Landau-Ginzburg arguments. Given the vector order parameter $\boldsymbol{\varphi}$, we write down the action in d spatial and one time dimension,

$$\mathcal{S}_{LG} = \int d^d r d\tau \left[\frac{1}{2} \left[(\partial_\tau \boldsymbol{\varphi})^2 + c^2 (\nabla \boldsymbol{\varphi})^2 + s \boldsymbol{\varphi}^2 \right] + \frac{u}{4} \left[\boldsymbol{\varphi}^2 \right]^2 \right], \tag{4}$$

as the simplest action expanded in gradients and powers of $\boldsymbol{\varphi}$ which is consistent

will all the symmetries of the lattice antiferromagnet. The transition is now tuned by varying $s \sim (\lambda - \lambda_c)$. Notice that this model is identical to the Landau-Ginzburg theory for the thermal phase transition in a $d+1$ dimensional ferromagnet, because time appears as just another dimension. As an example of the agreement: the critical exponent of the correlation length, ν, has the same value, $\nu = 0.711\ldots$, to three significant digits in a quantum Monte Carlo study of the coupled dimer antiferromagnet,[3] and in a 5-loop analysis[4] of the renormalization group fixed point of $\mathcal{S}_{LG}$ in $d = 2$. Similar excellent agreement is obtained for the double-layer antiferromagnet[5,6] and the coupled-plaquette antiferromagnet.[7]

In experiments, the best studied realization of the coupled-dimer antiferromagnet is $TlCuCl_3$. In this crystal, the dimers are coupled in all three spatial dimensions, and the transition from the dimerized state to the Néel state can be induced by application of pressure. Neutron scattering experiments by Ruegg and collaborators[8] have clearly observed the transformation in the excitation spectrum across the transition, as is described by a simple fluctuations analysis about the mean field saddle point of $\mathcal{S}_{LG}$. In the dimerized phase ($s > 0$), a triplet of gapped excitations is observed, corresponding to the three normal modes of $\boldsymbol{\varphi}$ oscillating about $\boldsymbol{\varphi} = 0$; as expected, this triplet gap vanishes upon approaching the quantum critical point. In a mean field analysis, valid for $d \geq 3$, the field theory in Eq. (4) has a triplet gap of $\sqrt{s}$. In the Néel phase, the neutron scattering detects 2 gapless spin waves, and one gapped longitudinal mode[9] (the gap to this longitudinal mode vanishes at the quantum critical point), as is expected from fluctuations in the inverted 'Mexican hat' potential of $\mathcal{S}_{LG}$ for $s < 0$. The longitudinal mode has a mean-field energy gap of $\sqrt{2|s|}$. These mean field predictions for the energy of the gapped modes on the two sides of the transition are tested in Fig. 2: the observations are in good agreement with the 1/2 exponent and the predicted[10] $\sqrt{2}$ ratio, providing a non-trival experimental test of the $\mathcal{S}_{LG}$ field theory.

4. Quantum "Disordering" Magnetic Order: Spinons and Visons

Now consider the triangular lattice antiferromagnet illustrated in Fig. 3, with nearest neighbor exchange constants J and J'. A fundamental difference from Section 3 is that now there is only one site per unit cell. Consequently, it is not as simple to write down a simple quantum "disordered' state, such as the large λ dimerized state in Fig. 1; any single pairing of the electrons into singlet bonds must break the lattice translations symmetry of the Hamiltonian, unlike the situation for the coupled dimer antiferromagnet. We will see that this difficulty leads to a great deal of complexity, and a rich class of field theories which can describe the quantum spin fluctuations.

A seemingly simple and well-posed problem is to describe the ground state of H_J for this lattice as a function of J'/J. However, the answer to this question is not known with anywhere close to the reliability of the model in Section 3. The main reason is that the sign problem prevents large scale Monte Carlo simulations,

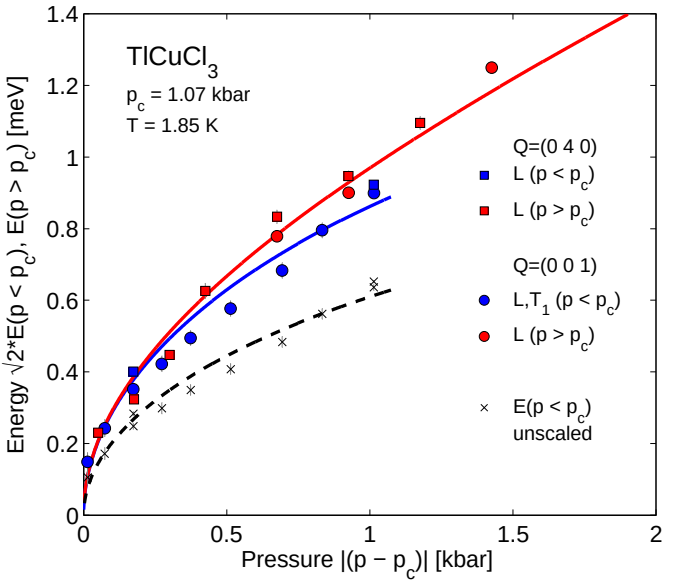

Fig. 2. Energies of the gapped collective modes across the pressure (p) tuned quantum phase transition in $TlCuCl_3$ observed by Ruegg *et al.*[8] We test the description by the action $\mathcal{S}_{LG}$ in Eq. (4) with $s \propto (p_c - p)$ by comparing $\sqrt{2}$ times the energy gap for $p < p_c$ with the energy of the longitudinal mode for $p > p_c$. The lines are the fits to a $\sqrt{|p - p_c|}$ dependence, testing the 1/2 exponent.

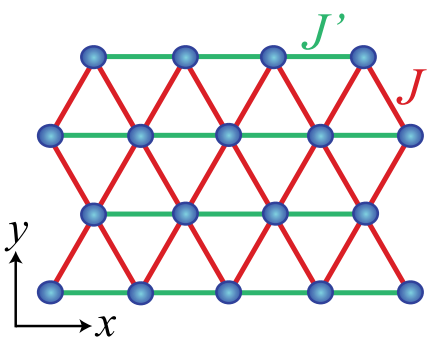

Fig. 3. The antiferromagnet on the distorted triangular lattice with exchange couplings J and J'.

and we have to rely on series expansions,[11] exact diagonalizations on relatively small systems,[12,13] or the recently developed variational approach based upon PEPS states.[14] Below, we will review theoretical proposals based upon an approach which begins from the ground state of the classical antiferromagnet, and attempts to quantum "disorder" it by a systematic analysis of the quantum fluctuations in its

vicinity.[15–18]

Experimental motivation for the antiferromagnet illustrated in Fig. 3 comes from a remarkable series of experiments by the group of Reizo Kato[19–25] on the organic Mott insulators $X[Pd(dmit)_2]_2$. These insulators crystallize in a layered structure, with each layer realizing a copy of the triangular lattice in Fig. 3. Each site of this lattice has a pair of $Pd(dmit)_2$ molecules carrying charge $-e$ and spin $S = 1/2$, which then interact antiferromagnetically with each other with exchange constants J and J'. The ingredient X intercalates between the triangular layers, and can range over a variety of monovalent cations. The choice of X yields a powerful experimental tuning knob, because different X correspond to different values of J'/J. The current status of experiments on these compounds is summarized in the phase diagram in Fig. 4. For small J'/J antiferromagnetic order is observed in NMR experiments. Indicated

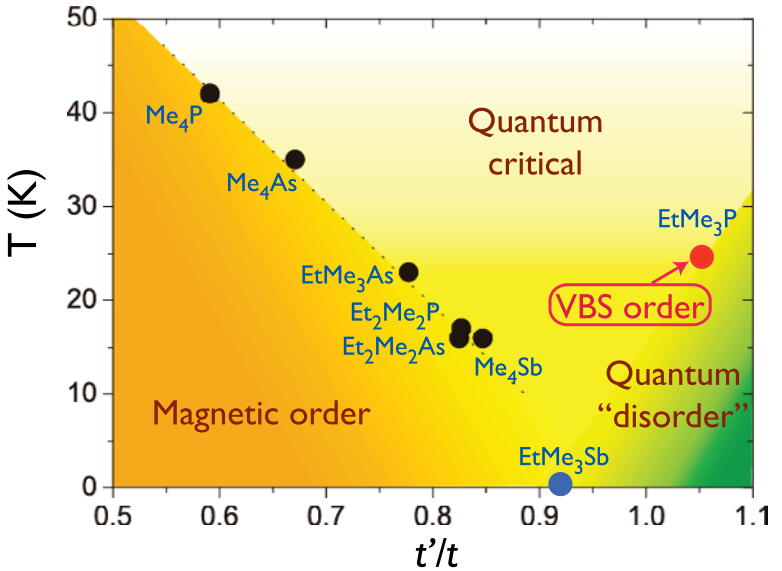

Fig. 4. Phase diagram of $X[Pd(dmit)_2]_2$ from Shimizu *et al.*[24] Each point is identified with the cation X. The values of the ratio of electron hopping, t'/t were obtained from quantum chemistry computatons; the exchange interactions $J'/J \approx (t'/t)^2$. The black points on the left represent compounds with antiferromagnetic order, and they are placed at the magnetic ordering temperature. The red point, $EtMe_3P$, is in antiferromagnet with a spin gap which acquires valence bond solid (VBS) order at the indicated temperature. Finally, the blue point, $EtMe_3Sb$, is a compound for which no order has been discovered so far.

in Fig. 4 is the magnetic ordering temperature: the long-range magnetic order is a consequence of the weak inter-layer coupling. For $J' = 0$, the lattice in Fig. 3 is equivalent to the $\lambda = 1$ square lattice in Fig. 1, and so the antiferromagnetic order is expected to have the two-sublattice Néel structure shown in the left panel of Fig. 1. It is evident from Fig. 4 that the strength of the antiferromagnetic order decreases

with increasing J'/J until ultimately yielding a quantum 'disordered' state. The nature of the latter state and of the quantum critical point to the magnetically ordered state are among the key issues we wish to address here.

Experiments on the $X[Pd(dmit)_2]_2$ Mott insulators indicate a possible structure of the quantum 'disordered' state. As indicated in Fig. 4, the compound with X=$EtMe_3P$ has valence bond solid (VBS) order in a state with a spin gap.[22,23] This is a state with an gap to all non-zero spin excitations of ≈ 40 K, as measured by an exponential suppression of the spin susceptibility. Below a temperature ≈ 26 K there is a doubling of the unit cell, consistent with the ordering of the singlet valence bonds as indicated in Fig. 5. Note that the wavefunction of this state appears

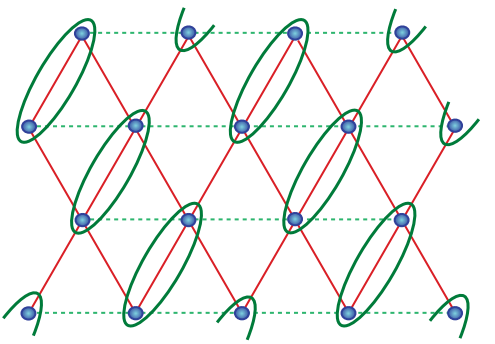

Fig. 5. Schematic of the valence bond solid (VBS) found in $X[Pd(dmit)_2]_2$ for X=$EtMe_3P$. The ellipses represent singlet bonds, as in Fig. 1.

similar to the coupled dimer state in the right panel of Fig. 1. However, the crucial difference is that the valence bond ordering pattern is not imposed by the Hamiltonian but is due to a spontaneously broken symmetry. There are 4 equivalent valence bond ordering patterns with an energy identical to the state in Fig. 5, obtained by operating on it by translational and rotational symmetries of the lattice. Theoretically, such a state was predicted[15] to exist proximate to the Néel state, with the symmetry breaking arising as a consequence of quantum Berry phases which are *not* present in the theory of the coupled-dimer antiferromagnet in Eq. (4). We note in passing that recent scanning tunneling experiments on the underdoped cuprates have also displayed evidence for VBS-like correlations.[26,27]

Also of interest in Fig. 4 is the compound with X=$EtMe_3Sb$ which has no apparent ordering[25] and so is in a "spin liquid" state. Its placement in the phase diagram in Fig. 4 indicates that an appropriate description might between in terms of a quantum critical point between the ordered phases. Its properties appear similar to another well-studied spin liquid compound, κ-$(ET)_2Cu_2(CN)_3$, which will be discussed in Section 7.1.

The subsections below will review the extensions needed to extend $\mathcal{S}_{LG}$ in Eq. (4) to be a complete theory of two-dimensional quantum antiferromagnets with a single

$S = 1/2$ spin per unit cell. A useful way of developing this extension is to postulate a spin liquid state in the form originally envisaged by Pauling[28] and Fazekas and Anderson:[29] a state which is a superposition of a large number of singlet bond pairings of the electrons (of which the pairing in Fig. 5 in just one) in a manner which preserves all the symmetries of the lattice. Such a state has two primary classes of excitations, spinons and visons, whose properties are reviewed below. As we will see, a rich variety of ordered phases and critical points are obtained when we allow one or more of these excitations to condense.

4.1. *Spinons*

Returning to our picture of quantum 'disordering' the Néel state, the key step[30] is to replace our vector order parameter $\boldsymbol{\varphi}$ by a two-component bosonic spinor z_α ($\alpha = \uparrow, \downarrow$)

$$\boldsymbol{\varphi} = z_\alpha^* \boldsymbol{\sigma}_{\alpha\beta} z_\beta, \tag{5}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices. We are clearly free to describe quantum fluctuations of the Néel order in terms of the complex doublet z_α rather than the real vector $\boldsymbol{\varphi}$. However, the new description is redundant: a spacetime-dependent U(1) gauge transformation

$$z_\alpha \to e^{i\phi} z_\alpha \tag{6}$$

leaves the observable $\boldsymbol{\varphi}$ invariant, and so should lead to a physically equivalent state. Any local effective action for the z_α must be invariant under this gauge transformation, and so we are led to introduce an 'emergent' U(1) gauge field a_μ to facilitate local gradient terms in such an action. This proliferation of degrees of freedom from the previous economical description in terms of $\boldsymbol{\varphi}$ might seem cumbersone, but it ultimately allows for the most efficient description of all the excitations, and their Berry phases.

Physically, the z_α operator creates a 'spinon' excitation above the spin liquid state. This is a charge neutral, spin $S = 1/2$ particle, represented by a single unpaired spin in a background sea of resonating valence bonds: see Fig. 6. In the formulation above, the spinon is a boson. Fermionic spinons are also possible, as we will discuss below in Section 5, and appear in the present approach as bound states of spinons and visons.[31–33]

4.2. *Visons*

Visons are spinless, chargeless, excitations of a wide class of spin liquids. They can be viewed as the 'dark matter' of condensed matter physics, being very hard to detect experimentally despite (in many cases) carrying the majority of the entropy and excitation energy.[34] They play a crucial role in delineating the structure of the excitations and phase diagram of quantum antiferromagnets.

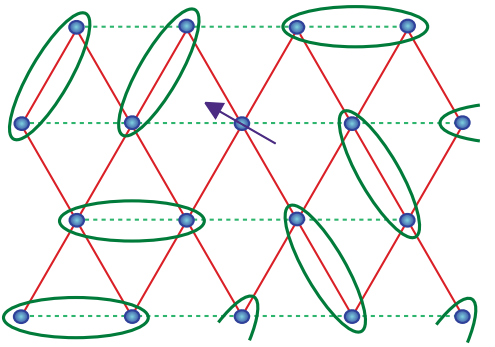

Fig. 6. Schematic of a spinon excitation. The unpaired spin hops on the sites in a momentum eigenstate, while the valence bonds resonate among many configurations.

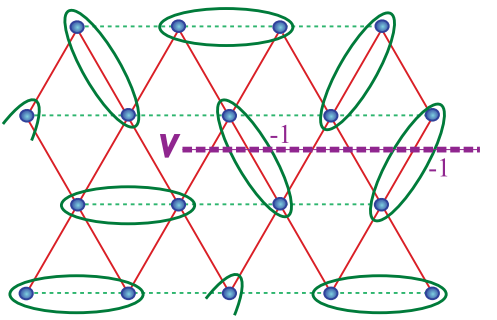

Fig. 7. Schematic of a vison excitation. It is a vortex-like excitation in the spin liquid, created by inserting the indicated phase factors for each configuration of valence bonds.

At the simplest level, a vison can be described[16,33] by the caricature of a wavefunction in Fig. 7. We choose an arbitrary 'branch cut' extending from the center of a vison out to infinity, and insert a factor of (-1) for each valence bond intersecting the branch cut. This yields a topological vortex-like excitation above the ground state. The motion of the point V in Fig. 7 in a momentum eigenstate yields a particle, which we represent by a real field v.

Note that the particle v is located on the lattice dual to the spins. An important aspect[35,36] of its motion on the dual lattice is that it is moving in an average background flux: just as vortices in a superfluid experience the background matter as an effective magnetic field (which is responsible for the magnus force), so do visons experience a net flux of π per direct lattice site containing a $S = 1/2$ degree of freedom in the underlying antiferromagnet. So we have to diagonalize the Hofstadter Hamiltonian of particles hopping on a lattice with flux to obtain the proper vison eigenstates. This straightforward procedure has two important consequences:[35–38]
(i) The vison spectrum has a non-trivial degeneracy tied to the flux per unit cell. We denote the degenerate vison species by fields v_a, where the index a ranges over

$1 \ldots N_v$, where $N_v \geq 2$ is the vison degeneracy.
(*ii*) The vison eigenstates have non-trivial transformation properties under the space group of the lattice in Fig. 3. This transformation is connected to the structure of the wavefunction of the Hofstadter Hamiltonian and will be important later in determining the nature of the phases proximate to the spin liquid.

4.3. *Solvable model*

Before writing down the field theory of the spinons, z_α, and visons v_a of the $S = 1/2$ Heisenberg spin model discussed above, we take a detour to describe an instructive exactly solvable model. This model was introduced by Kitaev,[39] and is the simplest Hamiltonian containing spinon and vison-like degrees of freedom. The solution of this model will make it clear that the spinons and visons are the electric and magnetic charges of an underlying gauge theory.

The Kitaev Hamiltonian can be written as

$$H_K = -J_1 \sum_i A_i - J_2 \sum_p F_p \tag{7}$$

where i and p extend over the sites and plaquettes of the square lattice, and $J_{1,2}$ are positive coupling constants. The operators A_i and F_p are defined in terms $S = 1/2$ Pauli spin operators $\boldsymbol{\sigma}_\ell$ which resides on the links, ℓ, of the square lattice; see Fig. 8. We have

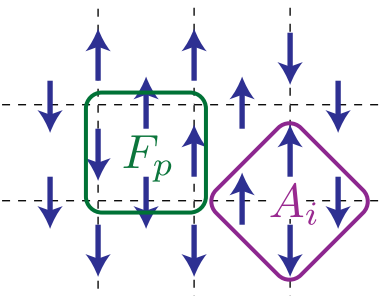

Fig. 8. The two terms in the H_K in Eq. (7).

$$A_i = \prod_{\ell \in \mathcal{N}(i)} \sigma_\ell^z \tag{8}$$

where $\mathcal{N}(i)$ extends over the 4 links which terminate on the site ℓ, and

$$F_p = \prod_{\ell \in p} \sigma_\ell^x \tag{9}$$

where now the product is over the 4 links which constitute the plaquette p. The key to the solvability of the Kitaev model is that these operators all commute with

each other, as is easily checked

$$[A_i, A_j] = [F_p, F_{p'}] = [A_i, F_p] = 0. \tag{10}$$

Despite these seemingly trivial relations, the eigenstates have quite an interesting structure, as we will see.

The ground state is the unique state in which all the $A_i = 1$ and the $F_p = 1$. Let us write this state in the basis of the σ^z_ℓ eigenstates. For $A_i = 1$ we need an even number of up (or down) spins on the 4 links connected to each site i. If we now color each link with an up spin, this means that the terms in the ground state consist only of closed loops of colored links: this is illustrated in Fig 9. Henceforth,

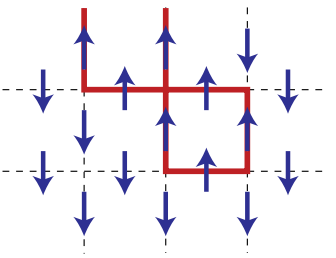

Fig. 9. A component of the ground state of H_K. The red lines connect up spins and form closed loops.

we will identify the states by their associated configuration of colored links. In this language, the action of F_p on a plaquette is to flip the color of each link in the plaquette; this has the consequence of moving, creating, and reconnecting the loops in the ground state, as illustrated in Fig. 10. It is now evident that to obtain a

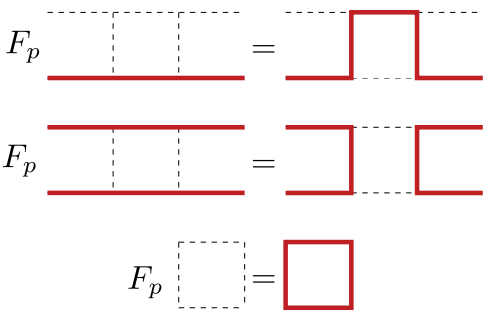

Fig. 10. Action of the F_p operator on sample configurations. Here p is the center plaquette of the left hand configurations.

state with eigenvalue $F_p = 1$ for all p, we should simply take the equal positive superposition of all closed loop configurations on the square lattice. This defines the spin liquid state in geometric terms.

Now let us describe the excited states. These turn out to be highly degenerate, an artifact of the solvable model.

The spinon excitation is a broken 'bond' in the σ^z_ℓ basis. (Because there is no conserved spin quantum number here, the spinon does not carry spin, but does disrupt the local exchange energy.) This is obtained by having a colored link end at a site i. The spinon state has the eigenvalues $A_i = -1$ and $A_j = 1$ for all $j \neq 1$. We still retain $F_p = 1$ on all plaquettes, and so the spinon state is the equal superposition of all loop configurations on the lattice with a single free end at site i, as shown in Fig. 11. If we interpret the Hamiltonian H_K as a Z_2 gauge theory, then the

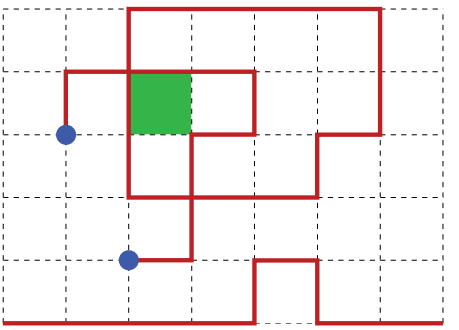

Fig. 11. A component of a state with 2 spinons and 1 vison. The spinons are the blue circles at which the red lines end. The vison is the green plaquette on which F_p has eigenvalue -1.

spinon carries a Z_2 electric charge.[16,40,41] Note also that the stationary spinon is an eigenstate, and so the spinon is infinitely massive. A generic model would have spinons moving in momentum eigenstates, with a finite mass determining the spinon dispersion.

The vison has the complementary structure. It has $F_p = -1$ on a single plaquette p, and $F_{p'} = 1$ for all $p' \neq p$. It returns $A_i = 1$ for all sites i, and so can be described geometrically by closed loop configurations. The wavefunction is still the superposition of all closed loop configurations, just as in the ground state. However, the signs of some of the terms have been flipped; Starting with the loop-free configuration, each time a loop moves across the plaquette p, we pick up a factor of -1 (see Fig. 11). In the Z_2 gauge theory language, the vison carries Z_2 magnetic flux.[16,35] Again, the stationary vison is an eigenstate, but a more realistic model will have a vison with a finite mass.

By choosing $A_i = \pm 1$ and $F_p = \pm 1$ we can now easily extend the above constructions to states with arbitrary numbers of spinons and visons. Indeed, these states span the entire Hilbert space of H_K. A typical state is sketched in Fig. 11. An examination of such states also reveals an important generic feature of the dynamics of spinons and visons: when a spinon is transported around a vison (or vice versa) the overall wavefunction picks up a phase factor of (-1). In other words, spinons

and visons are mutual *semions*.

Kitaev's construction generalizes to a large number of solvable models, some with much greater degrees of complexity.[42–50] The quasiparticles of these models carry electric and magnetic charges of a variety of gauge groups, and in some cases obey non-Abelian statistics.

4.4. *Field theory of spinons and visons*

Let us now return to the class of $S = 1/2$ Heisenberg antiferromagnets considered in Section 4. There are two key differences from the solvable Kitaev model: (*i*) the spinons, z_α, carry a global SU(2) spin label α, and (*ii*) the visons v_a have an additional flavor label, a, associated with their non-trivial transformation under the lattice space group. The visons of the Kitaev model do not have a flavor degeneracy because they do not move in an average background flux, a consequence of there being an even number of $S = 1/2$ spins per unit cell in this solvable model. However, the mutual semionic statistics of the spinons and visons does extend to the Heisenberg antiferromagnets, and can be implemented in a Chern-Simons field theory for its excitations.

In many of the interesting cases, including the lattice in Fig. 3, it is possible to combine the real vison fields v_a into complex pairs, and coupling the resulting fields consistently to a U(1) gauge field.[46,51–53] For the lattice in Fig. 3, the simplest possibility is[53] that there are only 2 vison fields, and these combine into a single complex vison $V = v_1 + iv_2$. Coupling this vison to a U(1) gauge field, b_μ, we then have the proposed field theory[53] for Heisenberg antiferromagnets on the lattice in Fig. 3

$$\begin{aligned}
\mathcal{S}_{zv} &= \int d^2r d\tau \Big\{ \mathcal{L}_z + \mathcal{L}_v + \mathcal{L}_{cs} \Big\} \\
\mathcal{L}_z &= |(\partial_\mu - ia_\mu) z_\alpha|^2 + s_z |z_\alpha|^2 + u_z (|z_\alpha|^2)^2 \\
\mathcal{L}_v &= |(\partial_\mu - ib_\mu) V|^2 + s_v |V|^2 + u_v |V|^4 \\
\mathcal{L}_{cs} &= \frac{ik}{2\pi} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu b_\lambda,
\end{aligned} \tag{11}$$

where μ is a spacetime index, and we need the Chern-Simons term $\mathcal{L}_{cs}$ at level $k = 2$. This field theory replaces the Landau-Ginzburg field theory $\mathcal{S}_{LG}$ in Eq. (4) for quantum 'disordering' magnetic order in antiferromagnets with one $S = 1/2$ spin per unit cell. Now we have two tuning parameters, s_z and s_v, and these yield a more complex phase diagram,[53] to be discussed shortly. We note in passing that the theory in Eq. (11) bears a striking resemblance to supersymmetric gauge theories[54] much studied in recent years because of their duality to M theory on $AdS_4 \times S^7/Z_k$: in both cases we have doubled Chern-Simons theories with bifundamental matter.

Apart from the usual Néel order parameter $\boldsymbol{\varphi}$ defined in Eq. (5), the presence of the vison field V allows us to characterize other types of broken symmetry. The space group transformation properties of V show that[37,38,53] V^2 is the VBS order parameter characterizing the broken lattice symmetry of the state in Fig. 5.

Secondly, in the phases where the vison V is gapped ($s_v > 0$), we can freely integrate the b_μ gauge field, and the Chern Simons term in Eq. (11) has the consequence of quenching a_μ to a Z_k gauge field. In such phases, the composite field $z_\alpha z_\beta$ is gauge invariant and can also be used to characterize broken symmetries; for the case being considered here, this order parameter characterizes an antiferromagnetic state with spiral spin order.

Using these considerations, the field theory in Eq. (11) leads to the schematic phase diagram[16,41,53] shown in Fig. 12. The phases are distinguished by whether

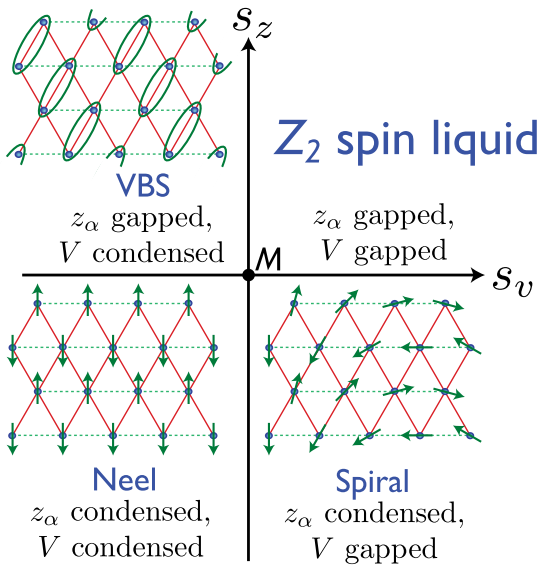

Fig. 12. A phase diagram for the Heisenberg antiferromagnet on the lattice in Fig. 3 obtained[53] from the field theory $\mathcal{S}_{zv}$ in Eq. (11). The same phase diagram was obtained[41] earlier by other methods.

one or more of the spinon and vison fields condense. Apart from the phases with broken symmetry which we have already mentioned (the Néel, spiral, and VBS states), it contains a spin liquid state with no broken symmetry. This is called a Z_2 spin liquid[16,40] because the spinons and visons carry only a Z_2 quantum number, a consequence of the arguments in the previous paragraph.

Note that this theoretical phase diagram contains the Néel and VBS states found in the experimental phase diagram in Fig. 4. It is possible that some of the compounds with magnetic order with larger values of J' actually have spiral order—neutron scattering experiments are needed to fully characterize the magnetic order.

The series expansion study of Weihong *et al.*[11] of the nearest neighbor antiferromagnet on the lattice in Fig. 3 finds the Néel, VBS, and spiral phases, and indicates that the point $J' = J$ is not too far from the multicritical point M in Fig. 12. This suggests that we analyze spin liquid compounds like X=$EtMe_3Sb$ in Fig. 4 by using

a field theory of quantum fluctuations close to M; this point of view is discussed further below in Section 7.1.

Another notable feature of Fig. 12 is the presence of a direct transition between phases which break distinct symmetries—this is the transition between the Néel and VBS states. Such a direct transition was discussed in some detail in early work,[15,55] where it was shown that the VBS order appeared as a consequence of Berry phases carried by hedgehog/monopole tunneling events which proliferated in the non-Néel phase. Here, we have given a different formulation of the same transition in which the Berry phases were associated instead with visons. VBS order has now been observed proximate to the Néel phase in numerical studies on a number of Heisenberg antiferromagnets on the square,[14,56–58] triangular,[11] checkerboard,[59–63] and honeycomb lattices.[64] We have also noted earlier the experimental detection of VBS-like correlations[26,27] in underdoped cuprates which are also proximate to the Néel state.

A direct second-order Néel-VBS transition is forbidden in the Landau-Ginzburg framework, except across a multicritical point. Arguing that transitions that violate this framework are possible at quantum critical points, Senthil *et al.*[65] proposed a field theory for the Néel-VBS transition based upon the Berry phase-induced suppression of monopoles at the critical point; so the criticality was expressed in a monopole-free theory.[66] In the approach reviewed above, this critical theory is obtained from $\mathcal{S}_{zv}$ in Eq. (11) by condensing V and ignoring the gauge field b_μ which is now 'Higgsed' by the V condensate; the resulting theory is $\mathcal{L}_z$, the CP^1 model of z_α and the U(1) gauge field a_μ. A number of large-scale computer studies[58,67–73] have examined the Néel-VBS transition. The results provide strong support for the suppression of monopoles near the transition, and for the conclusion that the CP^1 field theory properly captures the low energy excitations near the phase transition.[58,67,74] However, some simulations[69–71] present evidence for a weakly first-order transition, and this could presumably be a feature of a strong-coupling regime of the CP^1 field theory.

5. Spin Liquids near the Mott Transition

We now turn to the second route to exotic insulating states outlined in Section 2. Rather than using the insulating spin model in Eq. (2), we include the full Hilbert space of the Hubbard model in Eq. (1), and begin with a conventional metallic state with a Fermi surface. The idea is to turn up the value of U at an odd-integer filling of the elctrons so that there is a continuous transition to an insulator in which a 'ghost' Fermi surface of neutral fermionic excitations survives.[75]

This idea is implemented by writing the electron annihilation operator as a product of a charged boson, b, and a neutral spinful fermion f_α (the spinon)

$$c_\alpha = b f_\alpha. \tag{12}$$

Then we assert that with increasing U the boson undergoes a superfluid-to-Mott insulator transition just as in a Bose Hubbard model; this is possible because the

bosons are spinless and at an odd-integer filling. The superfluid phase of the bosons is actually a metallic Fermi liquid state for the physical electrons: we see this from Eq. (12), where by replacing b by its c-number expectation value $\langle b \rangle$, the f_α acquire the same quantum numbers as the c_α electrons, and so the f_α Fermi surface describes a conventional metal. However, the Mott insulator for the bosons is also a Mott insulator for the electrons, with a gap to all charged excitations. Under suitable conditions, the f_α Fermi surface survives in this insulator, and describe a continuum a gapless, neutral spin excitations—this is the spinon Fermi surface.

The most complete study of such a transition has been carried out on the honeycomb lattice at half-filling.[76,77] In this case the metallic state is actually a semi-metal because it only contains gapless electronic excitations at isolated Fermi points in the Brillouin zone (as in graphene). The electronic states near these Fermi points have a Dirac-like spectrum, and the use of a relativistic Dirac formalism facilitates the analysis. The low energy theory for the neutral Dirac spinons, Ψ in the insulating phase has the schematic form

$$\mathcal{S}_D = \int d^2 r d\tau \left\{ \overline{\Psi} \gamma^\mu (\partial_\mu - i a_\mu) \Psi \right\} \tag{13}$$

where γ^μ are the Dirac matrices in 2+1 dimensions, and a_μ is an emergent gauge field associated with gauge redundancy introduced by the decomposition in Eq. (12). Depending upon the details of the lattice implementation,[77] a_μ can be a U(1) or a SU(2) gauge field. For a large number of flavors, N_f, of the Dirac field (the value of N_f is determined by the number of Dirac points in the Brillouin zone), the action $\mathcal{S}_D$ is known to describe a conformal field theory (CFT). This is a scale-invariant, strongly interacting quantum state, with a power-law spectrum in all excitations, and no well-defined quasiparticles. In the present context, it has been labeled an algebraic spin liquid.[78]

Closely related algebraic spin liquids have also been discussed on the square[78–82] and kagome[83–85] lattices. In these cases, the bare lattice dispersion of the fermions does not lead to a Dirac spectrum. However, by allowing for non-zero average a_μ fluxes on the plaquettes, and optimizing these fluxes variationally, it is found that the resulting 'flux' states do often acquire a Dirac excitation spectrum.

One of the keys to the non-perturbative stability of these algebraic spin liquids[80–82] is the suppression of tunneling events associated with monopoles in the a_μ gauge field. This has so far only been established in the limit of large N_f. For the Néel-VBS transition discussed at the end of Section 4.4, there is now quite good evidence for the suppression of monopoles near the transition.[58,74] For the present fermionic algebraic spin liquids, the main numerical study is by Assaad[86] on the square lattice for antiferromagnets with global SU(N) symmetry; he finds evidence for an algebraic spin liquid for SU(4) but not for SU(2). The results of Alicea[82] indicate that single monopole tunnelling events are permitted on the square lattice, and these are quite likely to be relevant perturbations away from the fermionic algebraic spin liquid.

On the triangular lattice, an analysis along the above lines has been argued to

lead to a genuine Fermi surface of spinons.[87] In this case, the suppression of the monopoles is more robust,[81,88,89] but '$2k_F$' instabilities of the Fermi surface could lead to ordering at low temperatures.[90,91] A detailed study of the finite temperature crossovers near a postulated continuous metal-insulator transition has been provided for this case.[92]

An interesting recent numerical study[93] has presented evidence of the remnant of a spinon Fermi surface in spin ladders: the spectrum of a 'triangular' ladder contains excitations that can be identified with spinon Fermi points, and these can be regarded as remnants of a spinon Fermi surface after quantizing momenta by periodic boundary conditions in the transverse direction. It will be interesting to see if the number of such Fermi points increase as more legs are added to the ladder, as is expected in the evolution to a Fermi surface in two dimensions.

6. Exotic Metallic States

This section will consider extend the ideas of Section 4 from insulating to conducting states. This will yield metals with Fermi surfaces, some or all of whose quasiparticles do not have traditional charge $\pm e$ and spin $S = 1/2$ quantum numbers, and the temperature dependencies of various thermodynamic and transport co-efficients will differ from those in traditional Fermi liquid theory. In keeping with the unifying strategy outlined in Section 2, we will begin with a conventional Fermi liquid state, and induce strong quantum fluctuations in its characteristic 'quantum order'. We will consider the breakdown of Kondo screening in Section 6.1, and a quantum fluctuating metallic spin density wave in Section 6.2.

6.1. *Fractionalized Fermi liquids*

We begin with the heavy Fermi liquid state, usually obtained from the Kondo-Heisenberg model. The latter is derived from a two-orbital Hubbard model, H_U, in which the repulsive energy U associated with one of the orbitals (which usually models f orbitals in intermetallic compounds) is much larger than that is the second orbital (representing the conduction electrons). In such a situation, we can perform a canonical transformation to a reduced Hilbert space in which the charge on the f orbital is restricted to unity, and its residual spin degrees of freedom are represented by a $S = 1/2$ spin operator $\mathbf{S}_i$. These couple to each other and the conduction electrons in the Kondo-Heisenberg Hamiltonian

$$H_{KH} = \sum_{i<j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + \sum_k \varepsilon(k) c^\dagger_{k\alpha} c_{k\alpha} + \frac{J_K}{2} \sum_i \mathbf{S}_i \cdot c_{i\alpha} \sigma_{\alpha\beta} c_{i\beta} \tag{14}$$

where conduction electrons with dispersion $\varepsilon(k)$ are annihilated by $c_{k\alpha}$ and $c_{i\alpha}$ in momentum and real space respectively.

The heavy Fermi liquid state is obtained for $J_K \gg |J_{ij}|$ where the $\mathbf{S}_i$ are predominantly Kondo screened by formation of singlets with the conduction electrons. The structure of this state is most easily revealed by writing the local moments in

terms of neutral fermionic spinons $\mathbf{S}_i = f^\dagger_{i\alpha}\boldsymbol{\sigma}_{\alpha\beta}f_{i\beta}$. Then Kondo screening can be identified with the condensation of the bosonic field $B \sim f^\dagger_\alpha c_\alpha$. The f and c fermions strongly hybridize in the resulting state, leading to a "large" Fermi surface of the composite fermionic quasiparticle: the volume enclosed by this Fermi surface counts both the f and c fermions, and so obeys the traditional Luttinger rule. The absence of a bare hopping matrix element for the f fermions is responsible for the heavy mass of the quasiparticles at the Fermi surface.

Now consider increasing the values of the Heisenberg exchange interactions J_{ij}. We do this[94] with a set of J_{ij} which on their own would prefer to form one of the spin liquid states discussed in Section 4. Eventually, it will be preferable for the f moments to form singlets with each other, rather than being screened by the conduction electrons. This breakdown of Kondo screening happens at a sharp phase transition[95,96] in an effective gauge theory describing the disappearance of the Higgs condensate of B. Across the transition, we obtain a new non-Fermi liquid state labeled the 'fractionalized Fermi liquid'. The structure of this state is easily understood by adiabatic continuation from the $J_K = 0$ limit of H_{KH}: the f moments form one of the spin liquid states of Section 4, while the c conduction electrons form a Fermi surface of Landau-like, charge $\pm e$, spin $S = 1/2$ quasiparticles on their own. The unusual property of this decoupled limit violates the traditional Luttinger rule on the Fermi surface volume, which now includes only 'small' number of conduction electrons. The existence of this small Fermi surface is intimately linked to the presence of an exotic spin liquid on the f sites. It is the stability of this spin liquid which ensures adiabatic continuity, and retains the small Fermi volume even when J_K is non-zero.[95,96] The observable properties of this phase, of its transition to the heavy Fermi liquid state, and the connections to experiments on correlated electron compounds have been discussed elsewhere.[96–98]

6.2. *Algebraic charge liquids*

We turn here to the last of the methods noted in Section 2 for obtaining exotic states. We will begin with antiferromagnetically ordered state, but in a Fermi liquid, rather than in an insulator. It is traditional to refer to such ordered metals as spin density wave (SDW) states. Just as in the insulator, the SDW order can be characterized by a vector order parameter $\boldsymbol{\varphi}$. We can describe the loss of this SDW order in the conventional Landau-Ginzburg framework: this leads to a non-magnetic Fermi liquid state, and we will review this theory below. However, we wish to consider a possibility here which is the analog of that described in Section 4 for insulating antiferromagnets: we characterize the SDW order not by the vector $\boldsymbol{\varphi}$ but by the bosonic spinor z_α in Eq. (5). We wish to describe the loss of SDW order by a gauge theory of z_α fluctuations. This will lead to a non-Fermi liquid state[99–102] labeled an algebraic charge liquid (ACL). Just as was the case for insulators in Section 4.4, consistency of a z_α plus gauge field description of such a quantum critical point requires suppression of hedgehogs in the $\boldsymbol{\varphi}$ field (which are monopoles

in the a_μ field). This is facilitated here by the presence of Fermi surfaces which strongly suppress monopoles,[88,89] and we don't have to appeal to delicate Berry phase cancellations which were needed in the insulator.

It is useful to begin with a review of the Landau-Ginzburg approach to the loss of SDW order in a metal. The mean-field theory evolution of the Fermi surface with increasing SDW order is shown in Fig. 13, for a Fermi surface configuration appropriate for the hole-doped cuprates.[103,104] Right at the quantum critical point,

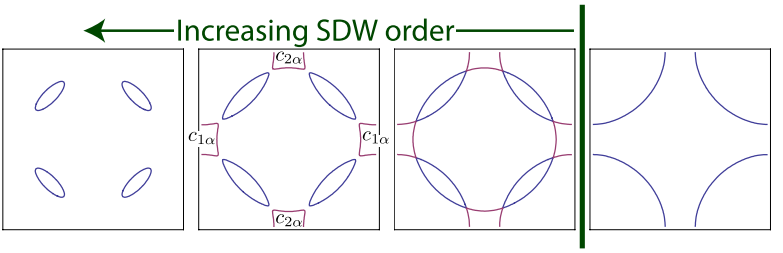

Fig. 13. Evolution of the Fermi surface of the hole doped cuprates in a conventional SDW theory as a function of the magnitude of the SDW order $|\boldsymbol{\varphi}|$. The right panel is the large Fermi surface state with no SDW order. The onset of SDW order induces the formation of electron (red) and hole (blue) pockets. With further increase of $|\boldsymbol{\varphi}|$, the electron pockets disappear and only hole pockets remain.

the SDW fluctuations, $\boldsymbol{\varphi}$ connect points on the SDW Fermi surface, and so can decay into a large density of states of particle-hole excitations. The damping induced by this particle-hole continuum modifies the effective action for $\boldsymbol{\varphi}$ from Eq. (4) by adding a strongly relevant term:[105,106]

$$\mathcal{S}_H = \mathcal{S}_{LG} + \int \frac{d^2k}{4\pi^2} \int \frac{d\omega}{2\pi} |\omega| |\boldsymbol{\varphi}(k,\omega)|^2 \tag{15}$$

where ω is a Matsubara imaginary frequency. The theoretical and experimental implications of this modified Landau-Ginzburg theory have been explored extensively in the literature. From Fig. 13 we see that this theory describes a quantum transition from a SDW state with electron and hole pockets, to a Fermi liquid state with a large Fermi surface.

Now we turn to the z_α description of the loss of SDW order. For this, we begin from the second panel in Fig. 13, the Fermi liquid SDW ordered state with electron and hole pockets. We introduce a parameterization of the electronic excitations in which their spin polarization is quantized along the direction of the local SDW order, determined by the local orientation of the z_α. Thus, let $c_{1\alpha}$ represent the fermionic quasiparticles in one of the electron pockets (see Fig. 13). We express these electrons in terms of fermions $g_\pm$ whose spin is polarized along the direction of the local SDW order:

$$\begin{pmatrix} c_{1\uparrow} \\ c_{1\downarrow} \end{pmatrix} = \mathcal{R}_z \begin{pmatrix} g_+ \\ g_- \end{pmatrix} \quad ; \quad \mathcal{R}_z \equiv \begin{pmatrix} z_\uparrow & -z_\downarrow^* \\ z_\downarrow & z_\uparrow^* \end{pmatrix} \tag{16}$$

For a uniformly polarized SDW state along the $\hat{z}$ direction, we have $(c_{1\uparrow}, c_{1\downarrow}) = (g_+, g_-)$, and so the $g_\pm$ are the usual up and down spin electron operators. However, Eq. (16) allows us to describe an arbitrary spacetime dependent orientation of the SDW order by using a 'rotating reference frame' defined by the SU(2) rotation matrix $\mathcal{R}_z$. A similar parameterization applies to fermions in the electron pocket which is shifted from the above pocket by the SDW ordering wavevector (π, π); we denote these fermions $c_{2\alpha}$ (see Fig. 13). For a uniform SDW order along the $\hat{z}$ direction, the Hartree-Fock theory of the mixing between the electron eigenstates now shows that $(c_{2\uparrow}, c_{2\downarrow}) = (g_+, -g_-)$; generalizing this to an arbitrary orientation as in Eq. (16), we have:[101,102]

$$\begin{pmatrix} c_{2\uparrow} \\ c_{2\downarrow} \end{pmatrix} = \mathcal{R}_z \begin{pmatrix} g_+ \\ -g_- \end{pmatrix}. \tag{17}$$

A similar analysis can also be carried out near the hole pockets.

We can now write down the general structure of the effective action controlling the z_α and $g_\pm$ modes. Within the SDW ordered phase with $\langle \varphi \rangle \neq 0$, this theory will be entirely equivalent to the conventional SDW Hartree-Fock theory, which is just re-expressed here in a different set of fields. However, it will lead to a new exotic ACL phase in the state without SDW order, $\langle \varphi \rangle = 0$. Constraints from gauge invariance and lattice symmetries lead to the action[99–102,108]

$$\mathcal{S}_{ACL} = \int d^2 r d\tau \Bigg\{ \mathcal{L}_z + g_+^\dagger \left(\frac{\partial}{\partial \tau} + i a_\tau - \frac{1}{2m} (\vec{\nabla} + i\vec{a})^2 - \mu \right) g_+ + g_-^\dagger \left(\frac{\partial}{\partial \tau} - i a_\tau - \frac{1}{2m} (\vec{\nabla} - i\vec{a})^2 - \mu \right) g_- \Bigg\}. \tag{18}$$

The spinon component, $\mathcal{L}_z$, of $\mathcal{S}_{ACL}$ is the same as that in Eq. (11), and the parameter s_z tunes the theory from the SDW phase with z_α condensed, to the ACL with z_α gapped. The $g_\pm$ fermions carry opposite charges under the a_μ gauge field, as is clear from the requirement that the $c_{1\alpha}$ and $c_{2\alpha}$ electron operators in Eqs. (16) and (17) be gauge invariant.

An analysis of the SDW-ACL critical point has been carried out using Eq. (18), and the result is relatively simple:[92,101] the $g_\pm$ fermions serve to damp the a_μ gauge field in a manner that it no longer couples efficiently to the z_α. At the critical point, the spin excitations are described by a theory of the z_α alone, and we can ignore their coupling to the gauge field and the fermions: consequently, the critical point is in the O(4) universality class, arising from the four real components of the z_α.

The structure of the ACL phase is also clear from Eq. (18). While the spinons are gapped, there are gapless charged excitations associated with Fermi surfaces of the spinless,[99–102,107–111] charge $-e$ carriers $g_\pm$. A detailed application of this structure to the peculiar properties of the underdoped cuprates has been discussed recently,[102] motivated by the evidence for the existence of electron pockets at high magnetic fields.[112] The attractive gauge force between the $g_\pm$ pockets causes them

to strongly pair into a s-wave paired state, with the local pairing $\langle g_+ g_- \rangle \neq 0$ and independent of momentum on the Fermi surface. Remarkably, application of the relations in Eqs. (16) and (17) shows that such a state actually corresponds to d-wave pairing of the physical electrons:[101] factorizing the expectation values of the fermions and bosons, we have

$$\langle c_{1\uparrow} c_{1\downarrow} \rangle = -\langle c_{2\uparrow} c_{2\downarrow} \rangle = \left\langle \left(|z_\uparrow|^2 + |z_\downarrow|^2 \right) \right\rangle \langle g_+ g_- \rangle \,. \tag{19}$$

The paired electron pockets also induce a weak proximity-pairing of the hole pockets, in a manner which is consistent with the overal d-wave pairing symmetry of the electrons; this leads to gapless nodal fermionic excitations along the Brillouin zone diagonals. These features have been proposed as an explanation of the 'nodal-antinodal dichotomy' observed in the underdoped cuprates.[102]

7. Experiments on Mott Insulators

This concluding section will highlight recent experiments on a variety of Mott insulators. We will complement the discussion by initially describing numerical studies of quantum spin models on the corresponding lattices. We will restrict our attention here to $S = 1/2$ antiferromagnets; there are also interesting examples of exotic states in the higher spin insulators $FeSc_2S_4$[113,114] and $NiGa_2S_4$.[115]

7.1. *Triangular lattice*

The most extensive numerical studies on the triangular lattice antiferromagnet have been carried out by the Paris group.[12,13,116] They have examined the phase diagram as a function of the ratio J_4/J_2, where J_2 is the conventional 2 spin exchange as in Eq. (2), and J_4 is the 4-spin ring exchange around all rhombi of the triangular lattice. For small J_4/J_2 there is convincing evidence[117] for 3-sublattice antiferromagnetic order (this is a commensurate version of the 'spiral' state in Fig. 12). For large J_4/J_2, there is reasonable evidence for a gapped Z_2 spin liquid state, similar to that discussed in Section 4.4, and in Fig. 12. The situation for intermediate J_4/J_2 seems unresolved at present system sizes. The theory of Fig. 12 would predict a direct second-order transition between these phases, but it has also been argued that there is an intermediate spinon Fermi surface state.[87]

Turning to experiments, the most detailed early experiments were carried out by Coldea and collaborators[118,119] on Cs_2CuCl_4. This compound has the geometry of Fig. 3, but with J'/J large. Notice that in the limit $J'/J \to \infty$, this lattice becomes equivalent to a set of decoupled one-dimensional spin chains. The neutron scattering experiments show spiral magnetic order, as is expected from the classical ground state for large J'/J. However, they also show an anomalous and strong continuum of non-spin-wave excitations at higher energies. Recent studies[120,121] have argued that these anomalous excitations can be quantitatively explained in a theory which begins with the decoupled spin chain solution and includes the effects of the interchain coupling, J, perturbatively.

We have already mentioned in Section 4 the extensive studies on the organic Mott insulators $X[Pd(dmit)_2]_2$ and their phase diagram in Fig. 4. A closely related set of experiments have been carried out by Kanoda and collaborators[122–124] on the organic compounds κ-$(ET)_2X$. So far, the Mott insulator has been studied only for $X = Cu_2(CN)_3$ and $X{=}Cu[N(CN)_2]Cl$ (unlike the many more examples in Fig. 4 for $X[Pd(dmit)_2]_2$). These compounds also have $S = 1/2$ moments on the lattice of Fig. 3. The compound $X{=}Cu[N(CN)_2]C$ has $J'/J \approx 0.5$, and, as can be expected by analogy from Fig. 4, it is clearly observed to have antiferromagnetic order. Much attention has focused recently on the compound with $X = Cu_2(CN)_3$ which is quite close to the isotropic limit $J'/J \approx 1$, and is a candidate spin liquid with no observed ordering at low T. Its properties are similar to the case $X{=}EtMe_3Sb$ in Fig. 4, and the analogy between these two series of compounds would suggest a description of κ-$(ET)_2$ $Cu_2(CN)_3$ using the proximity to the quantum phase transitions in Fig. 12; such a proposal has recently been examined in some detail.[34] Taken in isolation with studies of κ-$(ET)_2X$, such a proposal appears to require fine-tuning to place κ-$(ET)_2$ $Cu_2(CN)_3$ near a quantum phase transition, but the analogy with the phase diagram of the $X[Pd(dmit)_2]_2$ in Fig. 4 makes this proposal more natural. Also, we have noted earlier evidence from series expansion studies[11] that the isotropic point $J'/J = 1$ is not too far from the point M in Fig. 12.

Conflicting evidence on the nature of the low temperature state of κ-$(ET)_2$ $Cu_2(CN)_3$ has appeared in two recent experiments.[124,125] Specific heat measurements[124] show a non-zero value in the low temperature extrapolation of $\gamma = C_P/T$, which would be consistent with a spinon Fermi surface.[91] However, the extraction of the electronic specific heat requires the subtraction of a large nuclear contribution. This subtraction has been questioned by Yamashita *et al.*,[125] who instead measured the thermal conductivity, κ. This is not subject to contamination by a nuclear contribution, and they found a zero extrapolation of κ/T as $T \to 0$. Indeed, their low T behavior was consistent the thermal transport via a gapped electronic excitation. Qi *et al.*[34,53] have proposed identification of this gapped excitation with the vison, and have argued that this yields a consistent explanation of a variety of experiments.

7.2. *Kagome lattice*

The nearest neighbor antiferromagnet on the kagome lattice has been examined by a variety of numerical studies. The most recent evidence[126–129] points consistently to a ground state with a spin gap of $0.05J$ and VBS order. The pattern of the VBS order is quite complex, with a large unit cell, but was anticipated by Marston and Zeng[130] by an application of the VBS selection mechanism described in the $1/N$ expansion of the SU(N) antiferromagnet.[131]

Turning to experiments, recent examples of $S = 1/2$ kagome antiferromagnets are found in the compounds[132,133] $A_2Cu_3SnF_{12}$ with A=Cs and Rb. The A=Cs compound has the perfect kagome structure at room temperature, but undergoes a

structural transition at $T = 185$ K and details of the low temperature structure are not yet clear; the system orders magnetically $T = 20$ K. The A=Rb compound is a distorted kagome already at room temperature, which leads to distinct exchange interactions between nearest neighbor pairs with an average of $J \approx 200$K. This system does not order magnetically but has a spin gap of 21 K. It is tempting to associate the structural distortion and the spin gap with formation of the VBS state—a recent analysis along these lines has been provided by Yang and Kim.[134]

Volborthite $Cu_3V_2O_7(OH)_2{\cdot}2H_2O$ is another interesting kagome compound.[135–137] Here all the nearest neighbor exchange constants appear equal, and magnetic order of the spins is observed, albeit with a significant amount of spatial randomness.

Yamabe *et al.*[138] studied the compounds $Cs_2Cu_3MF_{12}$, with M=Zr and Hf, which form single crystal $S = 1/2$ kagome antiferromagnets with large exchange constants, $J = 360$ and 540 K respectively. These undergo structural transitions (to a not yet determined structure) at $T = 210$ and 175 K respectively. The low transition temperatures (compared to J) suggest that exchange interactions play a role here, and that there is connection between the structural transition and the physics of VBS ordering. At lower temperatures, magnetic ordering is observed at $T = 23.5$ and 24.5 K. Dzyaloshinsky-Moriya (DM) interactions are allowed on the kagome lattice,[139] and Yamabe *et al.* noted that these are likely the driving force for the magnetic ordering.

Finally, much attention has focused on the $S = 1/2$ compound herbertsmithite $ZnCu_3(OH)_6Cl_2$. This has $J \approx 170$ K and no observed ordering or structural distortion.[140–143] However, there is an appreciable amount of substitutional disorder between the Zn and Cu sites which affects the low T behavior.[144–147] More importantly, there is an upturn in the susceptibility at $T = 75$ K which has been ascribed to the DM interactions.[148–150]

Many of the above experiments indicate that analyses[151,152] of the effect of DM interactions on non-magnetic ground states of the Heisenberg Hamiltonian are needed for a complete understanding of the kagome antiferromagnet. For their Dirac algebraic spin liquid, Hermele *et al.*[85] showed that the DM coupling was a relevant perturbation, implying that an infinitesimal coupling induced magnetic order. In a recent exact diagonalization study, Cepas *et al.*[151] reach a different conclusion: they claim that there is a non-zero critical DM coupling D_c beyond which magnetic order is induced. They estimate $D_c/J \approx 0.1$, quite close to the value measured[149] for $ZnCu_3(OH)_6Cl_2$ which has $D/J \approx 0.08$. This proximity led Cepas *et al.* to suggest that the quantum criticality of the DM-induced transition to magnetic order controls the observable properties of this kagome antiferromagent.

7.3. *Hyperkagome lattice*

Okamoto *et al.*[153] have reported that $Na_4Ir_3O_8$ from a $S = 1/2$ antiferromagnet on a three-dimensional lattice of corner-sharing triangles, which they called the hyper-

kagome. This has no observed magnetic or structural ordering down to the lowest observed temperatures. Models of a spinon Fermi surface have been proposed,[154,155] but these overestimate the low T limit of the specific heat $\gamma = C_P/T$; present experiments do not indicate a significant γ in the low T limit. An analysis of a continuous Mott transition of this spinon Fermi surface state to a Fermi liquid has also been carried out.[156]

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Discussion

A. Georges Well, thank you Subir for this great talk, which covered a lot of topics. I would like to tell you how I was planning to organize the discussion. First, I think we could take about 10 minutes to take direct questions to Subir's talk. Then, I will say a few thing to try to set up the stage for the discussion, and then we will enter the discussion time. So, direct questions to Subir's talk right now. Chandra.

C. Varma It is really not a question, it is a comment on your amusing discussion about the Higgs. As you probably know, surprisingly enough, the theory for the Higgs particle for superconductors was not done until 1980. And the equivalent of the Higgs particle in the superconductors was experimentally identified in the period around 1980. So, the Higgs has already been....

S. Sachdev Yes, I apologize, there was very nice work of Chandra and others.

I should also mention the work of Maurice Rice and Ruth Norman who worked on this spin ladder compound model more directly and also talked about this longitudinal mode.

L. Glazman So, for the hidden Fermi surface effect. If I have a spin correlation function... So, this hidden spin order just means in spin-spin correlation function there are these oscillations underlying the Fermi vector..?

S. Sachdev Exactly. So, that is, for example, exactly what was measured in this numerical study of this model. It is just a strip, the triangular lattice strip, with this four-spin exchange term. And, this is the spin structure factor. So, take the spin-spin correlation function, take the Fourier transform, and you are getting various singularities here, and all of these wave vectors here, you can relate to sums and differences of the various ghost Fermi points.

L. Glazman So, in 1D, I mean, it is understandable in 1D, right? Just from the idea of bosonization of fermion. I guess it is a product of ....

S. Sachdev Yes... I agree it is less surprising in 1D. I guess Matthew will say more, Matthew is shaking his head, so please go ahead!

M. Fisher As soon as you have a two-leg ladder, the difference between bosons and fermions becomes real. So, I mean, just spinless bosons, hard-core, hopping in two-leg ladder with frustration. In the superfluid transition with one gapless mode, free fermions with two-leg ladder, you have two Fermi points. So, in this spin model, one is seeing signatures of those two Fermi points which would be on this zigzag chain that you have with small Hubbard interaction.

S. Sachdev But at the end of the day you can not understand the phases from bosonization approach, right?

L. Glazman Well, I would say no in this respect, say, in Lieb-Liniger model which is a one-dimensional bosonic model which has hidden order. So, if you look at correlation functions, they do oscillate at $2k_F$.

S. Sachdev I think this is a very nice way...

L. Glazman Oh, no.... it is more beautiful than any...

S. Sachdev Oh, yes, but it would of course be wonderful to answer this question in two dimensions more clearly. And we do not, you know.

M. Fisher No, I think the issue is, lets say, if you go to four legs zigzag. Do we see, you know, $2k_F$'s coming from four Fermi points, or the three legs zigzag, do we see $2k_F$ coming from three Fermi points?

L. Glazman And that is what happens there?

M. Fisher Well, we are not there yet but already we have gone from the one leg chain to two legs zigzag. And you have seen evidence of singularities in the spin correlation functions which are consistent with ghosts of two Fermi points.

L. Glazman So, your point is that the question becomes more interesting if you look at multi-channel case?

M. Fisher Well, as soon as you are out of strict, one channel, it is already quali-

tatively different.

L. Glazman OK, thanks because this clarifies....

S. Sachdev Well, I guess one other place where this is reasonably established that such a Fermi point exists is this kind of model here, ..., say, the Hubbard model on the honeycomb lattice but with electron spin with N components, where N is somewhat larger than two. So, at least for N finite but large, it is well established that such a phase exists. We would just like to know how long you can take N and under what conditions such a phase is going to survive. I think if you have a Fermi surface, yes, that has also been shown to be stable in the $1/N$ expansion.

A. Georges So, are there more direct questions to the speaker? Go ahead Seamus...

S. Davis Subir, can you give a prioritized list of things that you want experimentalists to detect about spin liquids?

S. Sachdev About spin liquids? It is a very difficult question. I do not have any easy answers, maybe others can try.

S. Davis Is that because you are assuming some things are undetectable or is it because there is a theoretical issue of defining key things to be detected...?

S. Sachdev No, it is closer to the first... there are certainly these topological excitations, these photons or these visons, they are definitely there. You have to distinguish them from ordinary phonons, for example. Maybe there is a way of doing that by just looking at quantum numbers. And you could couple to them by just... what will couple most efficiently to them is anything that changes the strength of the exchange energy, so that will couple to any of these particles. So the thing is that they may be there, they may be easy to detect. But saying for sure that is what you are seing is going to be hard.... And now, yes, I mean there is also this proposal of Senthil and Fisher and of mine where we talked about what will happen in a superconductor near the superfluid spin liquid state. Perhaps, there will be a remnant of this ordered state. But, there, it will be more dicy because you are going in a different phase and maybe you have lost everything, you know, you have to be really in the right condition to see that. It would be much nicer to see it directly in the spin liquid. So, for example, this κ/T we proposed in fact is dominated by the vison. It is not at all clear why that is correct... hum... but you know from that assumption we can try now to build a theory of experiment and see everything falls into place. But... I guess if you could find a way of looking down here and clearly distinguishing for example the phonon contribution to thermal conductivity from any electronic contribution, that would go a long way towards settling the issue here, for example.

A. Georges OK, I will take two more direct questions, then we will enter the discussion mode. There is Leon there.

L. Balents Actually, this is not so much of a question. Maybe responding a little bit to Seamus' question. At least for these Fermi surface states which

have a ghost Fermi surface, as was already been pointed out, they have characteristic oscillations in the spin correlations and, so it seems to me a reasonable smoking gun one might try to look for signature something like RKKY interaction in this system that is not a metal. This would be rather dramatic, and you can imagine experiments like looking for exchange biasing in multi-layers or some kind of local experiment which might also look for that. I do not think enough thinking along those lines have been done. Whether that thinking should be done by experimentalists or theorists or both is open.

P. Ong I would like to comment on the identification of (Louis) Taillefer's oscillations with electron pockets in cuprates. The evidence that these are electron pockets can be questioned. The thermal Hall effect which I take to be a much more reliable measurement of the quasiparticle sign is positive in this doping regime. But not up to 60 Tesla. That (experiment) has still to be done, but up to 50 Tesla, it is positive and shows no sign of going negative. Secondly, a worry is that the upper critical field in YBCO is astonishingly high. That has been shown in magnetization and Nernst effect experiments. I think that, even at 60 Tesla, the vortex contribution to the electrical current, the Hall effect, which is known to be negative, remains. I feel that dismissing entirely the condensate and its excitations is the weakness in (scenarios) claiming the electron pocket.

S. Sachdev Can I comment on that? I fully agree with the comments you are making. There are really two sides to these measurements at high field of 60 Tesla. One is the sign of the carriers and as P. Ong mentioned there is some ambiguity about that. But there is also these quantum oscillations that are periodic in $1/H$... and to me the only reasonable explanation for them is some kind of Fermi surface.... but whether electron-like or hole-like remains to be seen... and it is very hard to see how you get such well-formed oscillations. I do not have data here from a vortex-like picture as far as I am aware. Certainly this is a very hot field and there are many new experiments at 60 Tesla that are easy for theorists to throw out but that is one of the highest fields out there and I think there is going to be a lot of exciting advances from the high field measurements. And, yes, the period of these oscillations measures surprisingly the carrier density (hole or electron) of the order of the doping concentration, not the total density of the electrons. The simplest way to understand it is that you got some kind of broken symmetry and long-range orders but there could be more exotic reasons.

A. Georges OK, Z.-X., you want to say something?

Z.-X. Shen Yes, one follow-up to the question that Seamus has raised, what about experimental detection of the hidden Fermi point ? Would the momentum profile also have a singular behaviour so that if one measures the momentum profile would that allow you to see the hidden Fermi points in those systems?

A. Georges I think we can keep this issue actually for the discussion part. You may want to say more at that point.

Z.-X. Shen OK.

A. Georges Chandra, you wanted to react directly on what Subir were saying?

C. Varma Yes, I just wanted to briefly, since this issue has come out about this magneto-oscillation... Just wanted to remind you that the graphene which we are sure has Fermi points also has magneto-oscillations which are periodic in $1/H$.

S. Sachdev Well, that is fine. That is a Fermi liquid state from a metallic state.

C. Varma No, what I am trying to say is that if the cuprate underdoped state in the absence of superconductivity had a ground-state with four Fermi points, it would have magneto-oscillations periodic in $1/H$. There would be other experimental signatures of it being different like there are in graphene. For instance, if you go to infrared absorption in the magnetic field, the cyclotron energy would not be proportional to H but to H to some power less than one, but the magneto-oscillations would be showing a small... if interpreted conventionly it would be periodic in $1/H$ and would show a small Fermi surface.

S. Sachdev So, in what sense ... OK, you are talking about the pocket forming. So, as long as there is a nodal point, the Dirac part is not due to superconductivity, I agree with you.

C. Varma Yes, that is what I say.

A. Georges I think this question of quantum oscillations is going to be a major topic for discussion anyway. So, at this stage, I suppose we will return to that and let us thank Subir for his talk again.

A. Georges So now I would like us to really enter some discussion mode, as open as possible and just to set up the stage let me tell you how I view this. First of all, according to Bert's rule we should try to be as little late as possible. I know people prepared some transparencies: if you really want to use them, use them, but if you can dispense of using them, please do so! I am going to try to say things without any of that stuff. So what are we entitled to cover in this session? I would like to point your attention to the fact that there are two key words in the title of the session: one is *exotic* , whatever that means, and the other one is *models*.

What do we mean by "exotic" ? In my view, anything that has not or has rarely been seen before. Anything that is both poorly understood and potentially important: there are things that are poorly understood and not interesting so probably we should not count that as exotic. Or simply anything that we do not find yet boring, which perhaps is the best definition.

What do we mean by "models" ? Of course we mean theoretical models first, so these are things that do not exist but perhaps are relevant for something. We also mean, not only theoretical models, but real systems, real materials, which however are simple or clean enough so that some fun-

damental physics issues can be addressed by looking at them theoretically or experimentally. So experiments are not excluded from that session, as we have seen already. And we may also mean engineered systems, not only the materials of the chemists, but engineered systems like the ones in the mesoscopics or cold atoms, that is 'artificial' materials. And we should keep in mind that tomorrow we are going to have a session chaired by Maurice in which Catherine Kallin will be the rapporteur so there is some potential overlap on certain topics so keep this in mind for your contributions.

So we have a broad range of topics. We have heard a lot about quantum spins, quantum spin liquids, quantum spin transitions, this is great for the discussion, but you may also want to have moving electrons, they actually move in many materials and why only electrons? Actually you may want to have moving bosons, at the end of Subir's talk we had moving bosons. Besides that you may want to go beyond spins and consider orbital degrees of freedom. There are many important issues associated with the potential interest of orbital degrees of freedom. And then of course we may want to say things about current theoretical methods either numerical or analytical. Do we have the tools to address the interesting questions ?

So what are the questions? Well it's really for you to decide. I just want to emphasize a few of them. Spin liquids and associated quantum critical points: which are the established examples? Numerical evidence for models, again either theoretical models, or materials that have definitely a spin liquid state as an experimental realization. Orbital degrees of freedom: which new aspect do they bring? What about spin-orbital liquids? Can we actually use the orbital degrees of freedom to induce a disordered state ? And can we beat the crystal field to do that? Because, of course, if crystal field lifts all the degeneracies we are not going to be able to reach those states. Moving electrons or moving bosons: there are many issues there. Do charge fluctuations help stabilize unconventional phases? Continuous Mott transitions – that was also in Subir's talk – to Mott insulators, with or without broken symmetries.

And this brings us to the 2D Hubbard model and extensions with many questions, there will be overlap with tomorrow's session about that. Does it display momentum differentiation (nodes vs. antinodes) ? What is the relation with d-wave superconductivity ? There is this ghost Fermi surface that Subir talked about. And if yes does that momentum space differentiation correspond to just the fact that the system has not reached coherence everywhere in momentum space or does it truly correspond to a zero temperature fixed points of a different nature ? Then there are some questions that we may want to think about also, like 'can exotic states be used for something' ? For example in information processing. And finally: do they have interesting out-of-equilibrium dynamics ? Since we know there are many of them which come from frustrated systems with many competing

states. Ok, I am just throwing a bunch of issues, and now I would like to start really the discussion. So please, whoever wants to say something about the first aspect: spin-liquids, associated quantum critical points, numerical or experimental evidences. Patrick, are you showing data?

P. Lee – prepared comment On *Fermionic spinons.*

A. Georges Can I ask something ? How important are the charge fluctuations in your opinion ?

P. Lee It could be very important. I think one message is that the nearest-neighbor Heisenberg model does not describe these materials, at least in the organic case, and I emphasize that there are maybe ring exchange and also that there could be charge fluctuations. But morally it is still a spin-liquid: as long as it is an insulator and it does not order, then it is a spin-liquid.

P. Lee (continues his "Prepared comment" on *Dirac spinons and algebraic spin liquids*).

B. Altshuler Patrick, tell me if I have misunderstood you. What you are telling is that in this insulator, there are spinons, but then they will be paired because otherwise you would not have this effect? Now, there are spinons, are they different from usual spins...? To me pairing of spinons means that the spin-liquid does not survive...

P. Lee No, spinons are fermions which carry spin 1/2 excitations. Even if you pair them you still have Bogoliubov quasiparticles. Right? And depending on how you pair them, if you make a d-wave or f-wave, then you could still have gapless spinons, right? So they can survive. And the spin-liquid survives. No, making this spinon pairing does not give you long range order. So spin is still not long-range ordered. And you still have exotic excitations such as spinons and visons left. So the difference is that we converted the photon, it becomes gapped. That is the difference.

B. Altshuler But how do we see from this data that this pairing is not just destruction of spin-liquid?

P. Lee Oh because we know experimentally that there is no order. Down to 30 mK. And this is detected by NMR and USR. They are very precise. I think they can put very tight bound on the magnetic order.

P. Lee (continues his "Prepared comment"): So experimentally what happens is that there is this peak in the specific heat which suggests either some transition or some crossover. We do not know the nature of this.

B. Altshuler And this is done all at very low temperatures... there is no symmetry breaking to lowest...

P. Lee 30 mK.... and Yang Lee looked carefully at the kagome lattice and looked at lattice distorsions. You know, if you have a large unit cell ground-state, such as the ones Subir showed, you might suspect something like a lattice... and lattice are much more sensitive... he did not find any.

A. Auerbach Just a technical question about the susceptibility. You would expect the susceptibility in your spinon model to go to be a constant at zero

temperature or go to zero as a function of temperature.

P. Lee Oh, you mean that the Fermi surface will make it go to a constant, of course. Yes. If there is singlet pairing you may expect it to come down.

A. Auerbach And what do experiments say?

P. Lee In experiments it comes down below $5K$. Above $5K$ it is more or less constant and then it comes down by a factor of two, but still goes to a finite value. So there is a question: what does it mean ? So actually we made up a rather crazy story. Our pairing is of a rather exotic kind.

E. Shimshoni So I wanted to ask, there are very nice experiments on spin chains in quasi-one-dimension and spin systems measuring thermal conductivity which show very clear indication of Fermi surface for spinons controlled by a magnetic field. Can you do this for these materials...?

P. Lee Yes, so there is some data ... this thermal conductivity goes up a little bit, not very much , just a small amount with a magnetic field.

E. Shimshoni So what is the order of the exchange coupling?

P. Lee The exchange coupling is about 200 K. Very high.

E. Shimshoni So it is comparable to the phonon...

P. Lee Yes, the exchange is very high ...

E. Shimshoni But if you can look at spin systems with exchange coupling very different from phonon, from the Debye temperature, you could see a much clearer signature of the...

P. Lee Yes, you also would like to have as large an exchange coupling as possible because then you can go much below the conventional pairing temperature. I guess we do not want to talk about one-dimension, because one-dimension is well understood and it is a different class.

A. Georges It is no longer exotic!

P. Lee It is no longer exotic!

E. Shimshoni But it is the case where people have measured this...

P. Lee Yes I should take a look at those data, thank you...

A. Georges So, I would like to insist again that this should be a lively discussion. And we will start with some provocative things. There are various people who think that we should phrase more clearly key issues on spin liquids, and I am going to give free speech to people with that opinion. So, perhaps, Matthew first.

Prepared comment by Matthew Fisher[a]: Spin, Bose and Non-Fermi Liquid Metals in Two-Dimensions: Accessing via Multi-leg Ladders

1. Challenges in Mott Materials

The quantum theory of metals which identifies the Landau quasiparticles formed out of Bloch electrons as the appropriate independent electron-like excitations is a hallmark of 20th century physics. Despite its remarkable successes in describing many (weakly correlated) materials, it is now clear that this approach can fail qualitatively when the interactions are strong, as often occurs in materials with narrow partially filled bands originating from well-localized atomic d- and f-shells. Most dramatic are materials with a half-filled band that are Mott insulators because of the strong local Coulomb repulsion. An intriguing possibility is that such Mott insulators can exhibit exotic spin liquid ground states, having no magnetic or any other order.[1,2] A recent breakthrough is the appearance of several experimental realizations of spin liquids which all appear to be gapless, most notably some transition metal (d-shell) Kagome based crystals and a class of crystalline organic Mott insulators. The heavy fermion materials and cuprate superconductors are itinerant electron system which also appear to fall outside the rubric of the conventional theory of metals.

Many models have been proposed to understand these systems, such as Hubbard, $t-J$, and Kondo lattice models, but we essentially do not have controlled approaches to study them. On the analytical side, mean field treatments can look for states with broken symmetries, e.g. with spin or charge order, but cannot access non-Fermi-liquid physics. Among non-perturbative approaches, one is slave particle construction and gauge theory analysis while another is duality where one thinks in terms of topological defects like vortices; these approaches suggest the possibility of new phases in principle, but are uncontrolled for almost all 2D and 3D realistic models.

On the numerical side, Exact Diagonalization (ED) is limited to small systems, often too small to extract the physics. Quantum Monte Carlo suffers from sign problems. Variational Monte Carlo (VMC) calculations suffer from bias in the trial states. Dynamical Mean Field Theory does not capture all the important spatial correlation physics. Density Functional Theory, which is at the heart of realistic band structure calculations, describes well the "high-energy" (core) electrons, but does not capture properly the local Coulomb repulsion for the relevant electrons near the Fermi level. The Density Matrix Renormalization Group (DMRG) works extremely well in 1D, but capturing the entanglement inherent in strongly correlated 2D phases appears daunting.

[a]This text was co-authored by Olexei I. Motrunich and Donna N. Sheng

2. 2D Spin and Bose Metals

Many exotic spin liquid phases have been suggested by effective field theories (mostly gauge theory) and we now know that there are different kinds of spin liquids.[3] Gapped topological spin liquids are best understood and have been shown to exist in model systems.[4–7] Gapless spin liquids are also possible and will generically exhibit spin correlations that decay as a power law in space, perhaps with anomalous exponents, and which can oscillate at particular wavevectors. The location of these dominant singularities in momentum space provides a convenient characterization of gapless spin liquids. In the "algebraic" or "critical" spin liquids[3,8–10] these wavevectors are limited to a finite discrete set, often at high symmetry points in the Brillouin zone. But the singularities can occur along *surfaces* in momentum space, as they do in the Gutzwiller projected spinon Fermi sea state.[3,11,12] While the singular surfaces in such "quantum spin metals" are reminiscent of the Fermi surface in a 2D Fermi liquid, it must be stressed that it is the *spin* correlation functions that possess such singular surfaces – there are no Fermions in the theory – and the low energy excitations cannot be described in terms of weakly interacting quasiparticles.

There has been much less theoretical progress on non-FL conductors. Typically, the effective field theories have treated the electron charge sector as exhibiting conventional or classical physics. To explore the possibility of novel quantum behavior of itinerant charge carriers, two of us recently studied a closely related possibility of uncondensed but conducting quantum states of bosons.[13] This work proposed a 2D model of bosons with frustrating ring exchanges to realize a novel D-wave Bose Liquid (DBL), a "Bose metal" phase with low-energy excitations residing on "Bose surfaces" in momentum space. By combining with the spin sector, this can be extended to construct non-Fermi-Liquid electron states which have singular surfaces in momentum space that violate Luttinger's theorem. Other examples with critical surfaces have been studied recently.[14]

3. New Quasi-1D Approach to Spin and Bose Metals

Recently we argued that 2D spin metals, Bose metals, and non-Fermi-liquids phases which exhibit many low-energy excitations residing on surfaces in momentum space, should be accessible by systematically approaching 2D from a sequence of quasi-1D ladder models.[15] The ladder discretization of the transverse momentum cuts through the 2D surface, leading to a quasi-1D descendant state with a set of low-energy modes whose number grows with the number of legs and whose momenta are inherited from the 2D surfaces. These quasi-1D descendant states can be accessed in a controlled fashion by analyzing the 1D ladder models using numerical and analytic approaches, (ED, DMRG, VMC together with bosonization and gauge theory). These multi-mode quasi-1D liquids constitute a new and previously unanticipated class of quantum states interesting in their own right. But more importantly they carry a distinctive quasi-1D "fingerprint" of the parent 2D quantum fluid state.

The power of this approach was demonstrated in a recent paper[15] where we studied a new Boson-ring model on a two-leg ladder and mapped out the full phase diagram using DMRG and ED, supported by variational wavefunction and gauge theory analyses. Remarkably, even for a ladder with only 2-legs, we found compelling evidence for the quasi-1D descendant of the 2D DBL phase. This new quasi-1D quantum state possessed all of the expected signatures reflecting the parent 2D Bose surface.

It will be most interesting to search for analogous 2D spin metal phases in models possessing SU(2) spin symmetry. Particularly promising are "weak Mott insulators" which are located in close proximity to the metal-insulator transition, such as the organic triangular lattice Mott insulator κ-$(ET)_2Cu_2(CN)_3$ which appears to exhibit a spin liquid ground state. In these systems significant local charge fluctuations induce multi-spin ring exchange processes which tend to suppress magnetic or other types of ordering. Several authors have proposed that a mean field state with a Fermi surface of spinons is an appropriate starting point.[16–18] A preliminary analysis of the Heisenberg plus 4-site ring exchange spin Hamiltonian on the 2−leg triangular strip (using DMRG, ED, VMC and gauge theory) is indicating strong evidence for the anticipated ladder descendant of the spinon Fermi sea state over a large swath of the phase diagram.[19] It should be possible to extend this study to 3 and 4−leg triangular strips. One could also study the half-filled Hubbard model on triangular strips to see if the quantum state just on the insulating side of the Mott transition is descended from this 2D spin-metal phase. A 2D non-Fermi liquid phase of itinerant electrons that has singular surfaces which violate Luttinger's theorem (surfaces with the "wrong" volume, or perhaps even arcs), should also be accessible by systematically approaching 2D from a sequence of quasi-1D ladder models. In this case the momenta of the low energy quasi-1D modes will likewise violate Luttinger's theorem (which is valid for a "conventional" N−band Luttinger liquid).

Acknowledgments

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Discussion

A. Georges So, maybe, I can intervene here. Basically, what you are saying is that to stabilize these things you want to be close to the Mott transition and use partly the charge physics. Did I get this right?

M. Fisher Well, that is one way to stabilize them, to stabilize in this class.

A. Georges Is there any comment?

M. Fisher Another is to use strong frustration.

A. Georges I understand that, but that is one road.

M. Fisher Yes, that is one road.

A. Georges So, is there any comment from the audience about this? Do we have documented examples of being close to critical U and seeing some of that stuff happening?

M. Fisher ... There is this hyper-kagome materials in three dimensions, so...

A. Georges You want to say something about that?

T. Senthil I am just going to say that organics that have been discussed a lot are documented examples of systems that are close to the Mott transition.

A. Georges That is certainly true.

T. Senthil Empirically it is true. And likewise for the hyper-kagome lattice there is a recent experiment that also shows that, under pressure, it more or less undergoes the metal-insulator transition.

A. Georges Well, the fact that they are close to the Mott transition has been documented and the fact that we definitely have the spin liquid state... question mark ?

T. Senthil It is being discussed.

A. Georges It is being discussed, that is the correct statement.

M. Fisher Well, it is being discussed in these organic triangular lattice materials... but there is no evidence of any symmetry breaking.

A. Georges ... in some of them.

M. Fisher It is an insulator... and there is low energy saturation...

A. Georges Absolutely.

A. Auerbach I have a question of principle. If you say that the critical spin liquid is a phase, if it is critical then probably more than one susceptibility is infinite because it has power-law correlations in space. I would imagine unless you have long range interactions in the problem, you should also have power-law correlations in time, so why would you say it is robust against all perturbations and it is a stable phase?

M. Fisher I am not quite sure I will answer that question. I mean, ... For one thing, actually if it is a degenerate critical point whether or not there is a divergence in the susceptibility is not the same as whether or not there is a relevant perturbation at the critical point: It is empirical! so I think.... OK.

A. Georges More remarks? Sankar Das Sarma.

S. Das Sarma But Matthew, the question Assa just asked is an important question. I was just gonna ask the same question, because you have no gaps, so you are working with a model, you add some other term which is inherently present in a solid state system, take your organics. We need to know whether the phase survives. And I agree with you it can be done only numerically perhaps, but it is an important question. And, second, what is worrying me still is that the definition still involves what it is not, that it is not a broken symmetry state, so at low temperatures nothing happens. That is like a very dangerous way of looking for something, right?

M. Fisher Let me just say what protects a Fermi liquid from being unstable. Now ok, we know there is a BCS channel, ok and it is very low temperatures, we can see the susceptibilities... It is protected by phase space!

S. Das Sarma No, Fermi liquid is unstable. What are you talking about? Nothing protects you from having it.

M. Fisher No, it does not have to be.

S. Das Sarma We are lucky that the scale is very low, I mean, the scale is exponentially low.

A. Georges No, but there is a phase-space argument. He is right.

M. Fisher It is phase space, I mean, there is basically emergent symmetries there. I mean, in a Fermi liquid you know there is a conservation law that all quasiparticles at each patch on a Fermi surface are independently conserved. Why did that emerge? It emerged dynamically and it is an enhanced dynamical symmetry and it is protected by phase space to the extent that it is protected. Ok? There can be instabilities in the Fermi liquid, of course, this BCS instability. And you know, Patrick was alluding to that in this organic material where one has, at least in this description, a spinon Fermi surface that perhaps can be unstable to ...

S. Das Sarma That is a powerful analogy, I agree. But do you have a theorem like this here? Has any body proven the theorem?

M. Fisher Well, no.

S. Das Sarma Well, then, it is an analogy at the best. Right? I mean, I am being

contrarian just to be contrarian.

M. Fisher Well, this is a tough problem, Sankar. We can not solve it with free fermions, so we have to set the barrier lower towards what we mean by, you know,... an understanding as we develop it.

S. Das Sarma But the question of what perturbations do is an important question. Because if you are going to ask experimentalists to look for it, no experimentalist is going to have a system precisely following this model. And we have to tell them something and you know, I am not sure what to tell them.

A. Georges OK, Leon, you have an answer to what to tell them ?

L. Balents I just want to give a partial answer. At least I would not say that the instability is 100 percent firmly established but the things that you are specifically worrying about (operators with divergences in susceptibilities), those are operators which are not allowed by symmetry to be added to the Hamiltonian. For instance, they carry non-zero momentum, they are not translationally invariant or they break spin-rotation symmetry. You know, at any critical point, fixed point, usually protected by symmetry, the Ising critical point for example, you have to tune one parameter. But you would have to tune more if you break time reversal symmetry in magnets... So, the same story here. So, I would not worry too much.

A. Georges Nick Read.

N. Read As long as we are going into this much detail, I want to point out that you do not even have to have the symmetries to protect them. Whether a perturbation is relevant or not (and that causes the instability, Sankar), depends on what the value of the exponent is as well. And in the marginal case like BCS, you need the coefficients (the BCS coupling) and all the angular momentum channels to be negative and not positive, or positive and not negative, you know, to make it stable.

M. Fisher OK, I mean in a critical spin liquid, generically, there would be no reason to expect operators that are added to the Hamiltonian which are exactly marginal. You know that is a very special ...

A. Georges Boris has a question about your fruit tree.

B. Altshuler I am not an expert so excuse me if I am asking something stupid. I got used to the fact that if you have something critical, which means that there is no length scale, then there is some kind of conformal invariance in the problem. I mean, can you speak about this critical...?

M. Fisher No, I can not. Well, actually, ok, there are really two classes of critical spin liquids: those that are conformally invariant and that have, for example, correlation functions that would decay as power laws but would not oscillate in space. A Fermi liquid is not conformally invariant when you have a Fermi surface, but it is gapless and there are power law correlations...

A. Georges Well, Matthew, we will have to give the floor to other people...!

M. Fisher OK. Just one advertisement. Just as you can see evidence for a Fermi

surface in a model when you put on a two leg ladder, you can see evidence for these singular $2k_F$ surfaces in such critical spin liquid when you put them on a two leg ladder. So in principle and in practice, that is what we are finding. We gonna look at the realistic models, solve two, three, four leg-ladders using DMRG and see whether or not there are actually there... you know... in actual quasi-realistic models.

A. Georges OK, thanks, more people want to say something about key questions related to spin liquids? Mike Freedman, you wanted to say something?

Prepared comment by Michael Freedman: A Topological Phase in a Quantum Gravity Model

The concept of a topological phase may be traced to the interpretation[2] of the integer quantum Hall effect in terms of (what topologists call) a Chern class (or K-theory class) over the Brillion zone (momentum torus) and to Wilczek's realization[1] that in $(2+1)$-dimensions, anyon statistics was a possibility to be considered on an equal footing with the more familiar fermionic and bosonic statistics. The discovery of the fractional quantum Hall effect, Laughlin's wave function, charge fractionalization, and Halperin's realization[3] that the whole package must necessarily include anyonic statiatics, had by the mid 1980's presented us with a rather firm "existence proof" of topological phases. Unlike in mathematics, "proof" preceded definition. I would claim that two decades later we do not have a suitably general definition of what a topological phase is, or more importantly, any robust understanding of how to enter one even in the world of mathematical models. The latter is, of course, the more important issue and the main subject of this note. But a good definition can sharpen our thinking and a poor definition can misdirect us. I will not attempt a final answer here but merely comment on the strengths and weaknesses of possible definitions and argue for some flexibility. In particular, I describe a rather simple class of "quantum gravity" models which are neither lattice nor field theoretic but appear to contain strong candidates for topological phases. What is a topological phase? The easiest answer is that a topological phase is a system whose effective low energy theory is governed by a Chern-Simons Lagrangian. This answer is extremely efficient but limiting as it overlooks Dijkgraaf-Witten finite group[7] TQFTs which can be very interesting even in the absence of a Chern-Simons term (in this case a twist class $\beta \in H^4(BF;U(1))$) and possibly other, as yet unknown, topological structures. This definition would be a bit like defining a group to be a set of matrices with certain properties; the definition is too limiting since there are non-matrix groups. Similarly, any definition which contains phrases such as "spin-charge separation," "fractional charge," "point-like excitations," and "string-operator" presume too much: that electrons carry the relevant microscopic degrees of freedom or that the system is quasi $(2+1)$-dimensional, and may even unnecessarily exclude novel states of electrons confined in two dimensions. I prefer a spectral definition (but

will also criticize it!).

Definition 0.1. Let $\mathcal{H}$ be a Hilbert space with local degrees of freedom. A Hamiltonian $H : \mathcal{H} \to \mathcal{H}$ is said to describe a *topologial phase* if:

(1) H has degenerate ground state
(2) H has a gap to the first excited state
(3) (1) and (2) are "stable" with respect to any sum of local perturbations.

To explain: "gap" refers to a constant size energy gap in the thermodynamic limit and "stable" means, chiefly, that the splitting of the ground state degeneracy is exponentially small in a length scale (and also excludes the unlikely possibility that the gap closes instantly, in the thermodynamic limit, under perturbation). Equivalent to condition (3) is a more "cryptographic" condition (3'): for any local operator $\mathcal{O}$, the composition $G \xrightarrow{inc} \mathcal{H} \xrightarrow{\mathcal{O}} \mathcal{H} \xrightarrow{inc^\dagger} G$ is a scalar (or exponentially close to one). What I like about this definition is that is agnosist as to the type of local degrees of freedom, the dimension, and the nature and shape (e.g. could be string-like) of excitations. Also attractive is that equivalence classes of phases may simply be defined as the deformation classes of H subject to (1), (2), and (3). There are two things I do *not* like. First, to achieve a ground state degeneracy, periodic boundary conditions (e.g. wrap the quantum medium up to a closed surface) must be invoked. Since a topological phase is a *local* concept it is disconcerting to need a *global* ingredient in its definition. This state of affairs is like having a definition of hyperbolic geometry that did not work locally but only made sense for closed surfaces. (To keep the definition local, one might try using the existence of the constant $-\log \mathcal{D}$ term[4,6] in the von Neumann entropy of partial trace(ground state ψ_0), $S(P_A)$, but this approach has not yet been adequately explored.) Second, in the example, the "quantum gravity" Hamiltonian H_{qg} which I will now explain, will have gapless "gravity waves" which appear to have no interaction with the gapped topological degrees of freedom. Nonetheless, I would like to consider H_{qg} as defining a topological phase. Indeed, it appears to be the simplest route to realizing Turaev-Viro TQFTs (previously described in[5] using 12-body interactions). Thus, while definition 0.1 heads in the right direction, it is still too restrictive.

H_{qg} described for the $Dfib$ phase:

$H_{qg} = H^0_{qg} + \lambda V$ should be thought of as a bundle of Hamiltonians over the moduli space of metrics on a surface Σ (say a torus). The terms of H^0_{qg} are fusion constraints acting within fibers and F-moves which act between fibers. The Levin-Wen[5] 12-body plaquet term required to define the phase arises at second order from a perturbation λV which virtually excites an electric pair $(\tau \otimes 1, \tau \otimes 1)$ or $(1 \otimes \tau, 1 \otimes \tau)$ in the notation of.[7] We step back and specify the Hilbert space $\mathcal{H}$. $\mathcal{H}$ is spanned by kets which are pairs $|(\Delta, S)\rangle$, where Δ is a triangulation of n (fixed) triangles of the surface Σ and S is a labeling by particle types, in this case from the set $\{1, \tau\}$, of

the dual net N to Δ. We consider two triangulations Δ and Δ' (and their dual nets N and N') equivalent if they are isotopic on Σ (i.e. we can slide one onto the other). The dynamics on the set $\mathcal{N}_n$ of nets dual with n-vertex triangulations consist of the move in figure 1 and is known to mix algebraically $\lambda_1(\mathcal{N}_n) :: \frac{1}{n^\gamma}$, γ positive. Here, $\mathcal{N}_n$ is regarded as an abstract graph with vertices Δ_n and edges

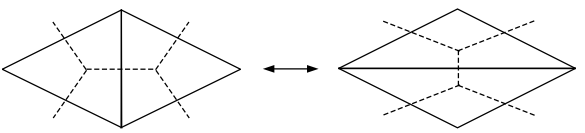

Fig. 1.

given by moves. λ_1 is the first eigenvalue of the graph Laplacian. To repeat, a ket is a net N with some edges marked 1 and some edges marked τ. N may be a regular honeycomb N^0, or a quite irregular N^i. Next we define H^0_{qg}. First, it enforces fusion rule terms at each vertex of each N^i by penalizing the illegal Fibonacci fusion (and its symmetries $\frac{1}{\tau} \succ 1$). Second, it contains terms between states of adjacent nets N and N' which enforce the unitary F-symbol $\begin{vmatrix} \tau^{-1} & \tau^{1/2} \\ \tau^{1/2} & -\tau^{-1} \end{vmatrix}$, $\tau = \frac{1+\sqrt{5}}{2}$. Let v, w be the normalized states of H, shown in figure 2. The second terms of H^0_{qg} are of the form $(id - |v\rangle\langle v|)$ and $(id - |w\rangle\langle w|)$. (In figure 2, solid lines carry the τ particle label and dotted lines the trivial label.) Finally, the perturbation λV creates an "electric

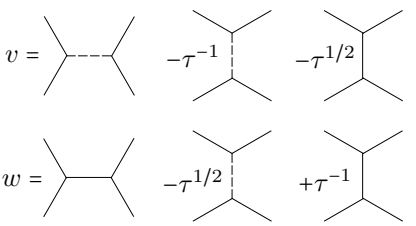

Fig. 2.

pair" (either $(\tau \otimes 1, \tau \otimes 1)$ or $(1 \otimes \tau, 1 \otimes \tau)$ by breaking a τ-labeled string resulting in a pair of excitations (fusion rule violations)[7]). $H_{qg} = H^0_{qg} + \lambda V$. Let us discuss the spectrum of H^0_{qg} first. H^0_{qg} is positive semi-definite and its ground state manifold consists of the states ψ with $\langle\psi|H^0_{qg}|\psi\rangle = 0$. Such a wave function ψ is completely determined by its restriction to a sample net N_0 via the F-symbols. (Importantly, ψ is not *over* determined (frustrated) since the F-symbol satisfies the pentagon equations.) The ground state manifold may be classified according to the number of magnetic particles $\tau \otimes \tau$ (of which, in our example system, there is only one type). Since we have only imposed fusion and F-moves there is no energy penalty for $\tau \otimes \tau$

charges, provided they are not frustrated and instead are allowed to roam ergodically according to the moves (F) which link adjacent nets. The magnetic charges on N^0 can return arbitrarily permuted, so the only zero energy (unfrustrated) states with j-magnetic charges, $j \geq 2$, are the ones that have equal amplitude for all positions of the j charges (on all n-vertex nets). One may think of the j-magnetic charge states as dispersing into momentum bands, although this terminology is not precise since translation does not even make sense on the general N_i. Nevertheless, it is true that for each $j \neq 1$, the j-magnetic charge states have four zero energy ("zero momentum") representatives $\{\psi_j\}$ on the torus. Above each of these are "bands" and gapless "gravity waves." The latter are "magnons" or phase oscillation across the (not very tightly bound) graph $\mathcal{N}_n$. Now consider a perturbation λV which

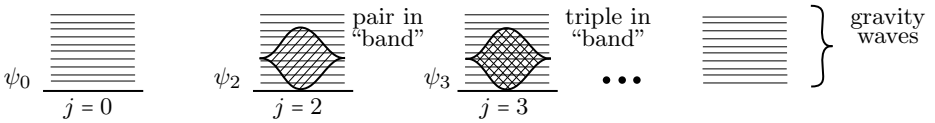

Fig. 3.

(virtually) pulls an electric pair (say $(\tau \otimes 1, \tau \otimes 1)$) out of the vacuum. Because of the *nontrivial* mutual statistics between the magnetic $(\tau \otimes \tau)$ and electric $(\tau \otimes 1)$ excitations, a frustration arises which increases the cost of the electric pair ψ_j^{e,e^*} in the presence of magnetic particles. For small j the effect is roughly linear:

$$\langle \psi_j^{e,e^*} | H_{qg}^0 | \psi_j^{e,e^*} \rangle - \langle \psi_0^{e,e^*} | H_{qg}^0 | \psi_0^{e,e^*} \rangle \approx j\alpha$$

for some $\alpha > 0$ and where we have set $\langle \psi_0^{e,e^*} | H_{qg}^0 | \psi_0^{e,e^*} \rangle = 1$. In the presence of λV a second order virtual process will lower the energy of ψ_j by $\sim \lambda^2 (1 + j\alpha)^{-1} \approx \lambda^2 (1 - j\alpha)$ producing an energy splitting separating the "true vacuum" ψ_0 from the "magnetically charged" vacuum by $\approx 2\lambda^2 \alpha$. ($\alpha$ is the energy scale of the F-symbol constraint divided by the square of the minimal number of moves (16) required to move a plaquet around a closed loop and across an "electric string" in any family of nets. This analysis is, so far, quite superficial. We should also consider the corresponding energy reductions induced by second order virtual processes between energy δ gravity wave states $\psi_{0,\delta}$ and $\psi_{j,\delta}$ above their respective vacua and the corresponding electric excitations $\psi_{0,\delta}^{e,e^*}$ and $\psi_{j,\delta}^{e,e*}$. However, the phase variations of any $\psi_{j,\delta}$, $j \geq 0$, over the number of moves (16) required to braid a $\tau \otimes \tau$ around electric strings can be made arbitrarily small by picking δ close to zero. Thus, the preceding argument adapts to show that the gravity wave states over ψ_0 are reduced in energy by this process more than the corresponding states over ψ_j, $j \geq 2$. (Details will appear in a joint paper with M. Troyer, whom I also thank for discussion on the concepts of this note.) Thus, the perturbation picks out the sector containing the true vacuum ψ_0 as lowest energy. A comparison of H_{qg} to the exactly solved Levin-Wen Hamiltonian H_{LW} is instructive. The ground states (in the thermodynamic limit) are expected to be bijective. The excitations of H_{qg}

are, in contrast to H_{LW}, mobile. To build point-like, confined excitations "wave packets" will need to be formed. Combinatorial recoupling arguments suggest that if such packets are confined in potential wells and braided, the L-W (i.e. Jones) braid representation should be exactly realized (in the thermodynamic limit). Thus, we expect that the entire topological structure, the TQFT, represented by H_{LW} is recaptured by H_{qg}.

H_{qg} is not a "lattice Hamiltonian." In particular, it is not defined on a "tensor product" Hilbert space (but rather a fiber-wise direct sum of these, one for each net in $\mathcal{N}_n$). Thus, it is not precise to assert that H_{qg} is "k-body" for any k, but it is evidently quite simple. One may say that the flux (plaquet) term of H_{LW} (which is 12-body) has been simulated by more local interactions, but to achieve this we have resorted to a context where the lattice itself fluctuates and must be counted among the dynamic variables. Hence the sobriquet: quantum gravity Hamiltonian.

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Discussion

A. Georges So, more on the no-lattice issue? Bert?

B. Halperin Well, I think I could consider two possibilities: one is where you allow these moves, you are sort of changing the linking of the lattice, you are not changing number of the lattice sites...

M. Freedman Yes.

B. Halperin And, if that happens only locally, some sort of virtual excitations, then probably Matthew would be happy, because then on average you would return to the original... I know it is risky for me to speak but I would distinguish a case where it is actually kind of melted...

M. Freedman Ah, yes.

B. Halperin ... and these dislocations, as I would call them, are flowing all over the place and you have broken the translation symmetry.

M. Freedman I agree with that distinction. I was thinking of the networks as just a mild high frequency perturbation around the flat metric, so you essentially still can move in a momentum basis.

A. Stern There are thousands of possibilities to change this kind of Wen-Levine or Freedman and company hamiltonian... Why should we expect this one, and

not another to give a topologically non-trivial phase with non-Abelian quasi-particles or something like that? Do you have any intuition?

M. Freedman I think that sort of the holy grail in the subject is to understand in generic terms what will lead to a topological state. You know, as I said, if we did not have the quantum Hall effect we would be very discouraged. We might not think they are easy to find, but if you think about the quantum Hall Hamiltonian, it is almost as simple as possible, well, at least the mathematical idealization of it, you know as a 2DEG with a transverse magnetic field and it apparently exhibits very interesting and presumably even non-Abelian topological phases. So, we have a great reason for optimism and we, you know, we should not be discouraged that the exactly solved models on the lattice that we see contain these very large twelve body interactions. That we should think positively and what I am trying to do is just describe another point of view that could lead to topological phase without such an elaborately fine-tuned interaction. And the $6j$-symbols are more elementary.

A. Stern Why do you expect this direction to ....

M. Freedman To work? Well, I think that one can prove the only gap would be in trusting the second-order perturbation theory but I think to physicists' satisfaction one can prove this exhibits exactly the same ground-states on, say, closed surfaces as Levin-Wen phase if you input identical fusion categories.

A. Kitaev I would like to say a few words, kind of speculative. We know that three dimensional models, critical three-dimensional models are generally not exactly solvable. However, one can imagine a situation where one has a model coupled to gravity, and that would make it exactly solvable. I do not know if such a model exists but it is an interesting possibility to consider it.

M. Freedman In 3+1...?

A. Kitaev No just three, or 2+1.

M. Freedman I see.

A. Georges More comments? OK, so I think there are still a couple of issues on spin liquids we wanted to deal with... Xiao-Gan Wen next.

X-G. Wen – prepared comment "What is a Spin liquid?"

A. Georges So you are trying to establish a dictionary between two languages and we understand neither of them. That is what you are trying to say.

X. Wen Yes, exactly. We do not have a dictionary.

X-G. Wen Goes on with his presentation.

A. Georges Who wants to take upon that? Patrick?

P. Lee I just want to remind you that we have a few experimentalists here. We should not totally discourage them. I am telling them that they are not going to see anything....

A. Georges Thank you, I was about to make that comment!

P. Lee You know there are actually a lot of experimental signatures that tell you that you have an exotic state such as, you know, linear specific heat, and I did not mention that the $1/T$ has power low down to very low temperature.

And, going back to the quantum Hall analogy, experimentalists had a lot of fun for many years, finding many new phenomena without knowing that there is a topological structure behind it.

X. Wen Actually, I do not mean that, wait. Although we do not have a general, universal measurement for the pattern of entanglement. However, once we have a material which we think is a spin liquid, we have some theory, then usually for a specific material we have clear cut.....

A. Georges Can you give us an example? Give us a concrete example.

X. Wen Like what Patrick mentioned, if you assume that it is a Dirac spin liquid, then, there is a prediction that in a magnetic field it would induce an order. With an order proportional to the magnetic field. So, usually, for the concrete spin liquid theory, there is a smoking gun concrete experimental measurement. We do not have a universal probe, which probes every spin liquid. But usually for each particular materials we have a concrete measurement, a concrete smoking gun proposal. So, it is not that bad.

B. Altshuler Xiao-Gang, may I ask you if you can give a kind of intuition about this topological protection. Let me tell you what I mean. We got used to the fact that in order to have gapless excitations you need..... first of all gapless excitations usually is at $k = 0$ – long wavelength. What you need to mean is some kind of global symmetry like you can shift the whole crystal as a whole and phonons have no gap, you can make global gauge transformations without paying anything, you have gapless phonons and so on. Can you give us this type of intuition , so why $k = 0$ excitations, ... because of topological protection have a zero energy?

X. Wen Yes, this is a little difficult. Let me say that the argument is the following. There is a model which has an emergent photon. And there is a kind of limit in the model where you can see clearly that there is a closed loop. The fluctuations of closed loops give you a gauge boson, U(1) boson. Then, the condition of this closed loop turns out to be the gauge symmetry. So, the gauge symmetry requires a closed loop. Then, you will say in a condensed matter system we can add a term to break the loop to have an open end! You will say that gauge symmetry is broken and the photon becomes massive. Well, that turns out not to be the case. There is another term. Well, then you have an open string. However, in a spacetime picture, the open string really means that you have a hole in a membrane. This hole can be repaired, by this I mean you can modify the gauge transformation a little bit and you will find that the whole system still has a modified gauge transformations. So, that is what we proved: the gauge symmetry is topological. The gauge symmetry means the closed loop condition, and this is topological. In a system where we add a perturbation, you cannot break the gauge symmetry for a weak perturbation and if you believe that the gauge symmetry.....

A. Georges Xiao-Gang, I have to cut you off...

X. Wen Yes, fine...

A. Georges ... and I am going to take up on Patrick's remark. Oh, Naoto wants to say something...

N. Nagaosa So I want to ask you about the exclusive nature of the long-range order and topological order. Is there any possibility to have the coexistence of the two orders?

X. Wen Yes, that is right...

A. Georges Let me just intervene; I would like to follow Patrick's suggestion that we should go back a little bit down to Earth, on to observable things. But please go ahead. Let us finish this first. Can you answer Naoto's question?

X. Wen Oh, yes, the answer is yes. So the long-range entanglement and the local order are totally consistent. Actually, that is what I emphasized. Using local order, however, is not an issue. One should not emphasize that. The key issue is: do we have a long-range entanglement? And the local order does commute with this picture and they do not interfere with each other.

A. Georges OK, so thank you Xiao-Gang. If you remember the various issues that I had pointed out at the beginning of the discussion, one way of connecting to possible observable things was to exploit orbital physics. And I am sure that some people can make comments on that, Leon has something to say. We may also hear some comments by Bernd Keimer.

Prepared Comment by Leon Balents: Some Promising Model Systems for Exotic Phenomena

In the context of this meeting, two meanings of model are appropriate: (1) an imaginary system or "model Hamiltonian" simplified so as to be suitable for theoretical study and (2) a physical system which is unusually simple and whose microscopic physics is sufficiently well understood that it provides a particularly useful venue for confronting theory and experiment. It is easy to find successful examples of the first type of model – the Sommerfeld model for metals, the Ising model for ferromagnetism and phase transitions, the Kondo model of local moments interactions, and the Luttinger model of one dimensional interacting electrons. In the second category of model systems we have liquid helium in both its isotopes, providing realizations of atomic and paired superfluids, and the best Landau Fermi liquid, and the two dimensional electron gas in GaAs/AlGaAs heterostructures, out of which arose the integer and fraction quantum Hall effects, much of mesoscopic physics including diverse weak localization phenomena, Coulomb blockade, etc., and other correlation effects such as Coulomb drag. I'd like to speculate here about some interesting directions for both sorts of models in the near future.

Let me first discuss model Hamiltonians. In the study of "exotic" phenomena, these have really taken on a life of their own. There are some good reasons for this. Probably the foremost is that until recently, it seemed possible that exotic states such as quantum spin liquids might be entirely artifacts of drastic approxi-

mations such as slave particle mean field theories. Studies of model Hamiltonians such as quantum dimer models have provided explicit realizations of such phases, proving that they are actually possible in non-pathological (though not very realistic) models. Broadly speaking, we use model Hamiltonians for two purposes: to study fundamental issues (such as in these proofs-of-principle), and to address experiments in a simplified venue. In the best cases, studies of model Hamiltonians combine both goals, as in the use of the BCS reduced Hamiltonian in the theory of superconductivity. I believe that the tie to experiment is essential, and in the long run it is only those theories which make this linkage that will have a lasting impact. Fortunately, there is no lack of interesting experiments where exotic phenomena can be explored in *appropriate* models guided by the physics.

One good place to look is in the problem of orbital physics in strongly correlated materials. For the most part, correlated electrons arise from tightly bound d or f atomic states. These often retain some degree of the hydrogenic orbital degeneracy even within a crystal. When such degenerate shells are not full, half-filled, or empty, electrons retain an orbital degree of freedom in addition to their intrinsic spin. Indeed, this situation is the rule rather than the exception, and it is well known that orbital physics is crucial to the vast majority of correlated transition metal oxides. A nice review and an indication of the richness of such phenomena can be found in Ref.[1] Orbitals introduce a number of new ingredients into models: strong directionality of hopping and exchange, interaction with Jahn-Teller phonons, and non-trivial spin-orbit coupling effects. While the prototypical Kugel-Khomskii model[17] of the orbital dynamics in certain Mott insulators was introduced already in 1972, this was just the tip of the iceberg. Experiments are *far* ahead of theory in this area, and the latter is dominated by *ab initio* calculations, which are least well suited for the most interesting problems involving strong interactions and fluctuations. Moreover, with recent progress in elastic and inelastic resonant x-ray scattering techniques promising direct measurements of orbital correlations, the time is ripe for fruitful interaction of correlated electron theory and experiment. Basic issues for theory include the existence and character of orbital and spin-orbital liquids (analogous to spin liquids), and nature and consequences of frustration of orbital interactions.

I'd like to more specifically mention a number of interesting problems coming into focus related to the combination of spin-orbit coupling and orbital physics. At the level of single-particle states, spin-orbit coupling can split otherwise degenerate orbital multiplets to form new states with strongly entangled spin and orbital components described by complex wavefunctions. These effects are strongest in 5d ions, with Ir^{4+} being an especially promising candidate. In Mott insulators, where Coulomb interactions are crucial, there are still dramatic effects. A recent preprint[2] proposes this as a novel mechanism to realize a time-reversal invariant "topological insulator", discussed elsewhere in this conference. In the material $Na_4Ir_3O_8$, which is one of best experimental quantum spin liquid candidates,[3,5,6] strong spin-orbit coupling has been argued to play an essential role.[4] A recent theoretical paper[7] pro-

poses a means to realize the Kitaev model[8] (of non-abelian anyons) and more general "quantum compass" models using spin-orbit effects. Even in 3d ions, if exchange is sufficiently weak, spin-orbit coupling can dominate the physics. For example, it has recently been proposed[9] that a spin-orbit driven quantum critical point is behind the anomalous "spin-orbital liquid" phase observed in the spinel $FeSc_2S_4$.[10] With all these recent developments, the confluence of spin-orbit physics, orbital degeneracy, and strong interactions seems a very promising area to search for novel effects. Perhaps the observed metal-insulator transitions in pyrochlore iridates[11] might be fertile ground for theorists?

Let's turn now to physical model systems. I think there is no doubt that semiconductor heterostructures (mostly the GaAs/AlGaAs variety) were the dominant model system for much of the last twenty years. Probably the key factors which contributed to this success were the clarity of the underlying Hamiltonian – to an excellent approximation, one could achieve a nearly ideal "Jellium" model of a two-dimensional electron gas – and their tunability and cleanliness. To some extent ultra-cold trapped atoms seem a close parallel, with immense tunability, no disorder, and well-understood atomic Hamiltonians. Those systems are covered in their own topic in this conference. There are, however, less well-known but very interesting new model systems arising in the solid state. An historically important one that is gaining new life lately is elemental bismuth, a semi-metal with astonishingly small Fermi volume of 10^{-5} of the Brillouin zone. The very low carrier density makes the ultraquantum (lowest Landau level) limit accessible at laboratory fields, and also makes a continuum approximation (similar to Jellium) possible. Unlike in semiconductor heterostructures, however, bismuth is fully three-dimensional. Recent experiments give evidence for very long mean free paths and clearly show collective phase transitions[12] and behavior reminiscent of the fractional quantum Hall effect.[13] Theoretically, three-dimensional electron systems in the ultraquantum limit are a *terra incognitia*, with understanding limited primarily to weak interactions. It is clear that anything similar to fractional quantum Hall physics in such a situation must involve radically new ingredients. With the continuum Hamiltonian of bismuth well understood, this is an attractive area for future study.

I'll finish by discussing a materials effort which is still in its infancy, but growing rapidly. Since 2002,[15] a growing number of groups have been pursuing the layer-by-layer growth of complex transition metal oxides by techniques similar to molecular beam epitaxy and pulsed laser deposition as used in semiconductors. Ultimately, this might provide the ability to make the same sorts of structures as in current semiconductors (2DEGs, quantum dots, wires, etc.) but with the added functionalities of correlated electrons such as magnetism and superconductivity. In the near term, these efforts provide a new degree of control to investigate and perhaps engineer correlation phenomena. Perhaps most exciting is the chance to manipulate the orbital state of transition metal ions at interfaces and in quantum wells, through strain, changes in Jahn-Teller effects, crystal fields, and covalent bonding. For instance, a recent experiment demonstrated a transfer of the hole from the usual

$d_{x^2-y^2}$ orbital to the d_{z^2} orbital in the high-T_c superconductor YBCO near an interface with LCMO (a ferromagnetic manganite).[14] It is also possible to create a large (2d) charge density near an interface in effect "doping" a Mott insulator in such structures.[15,16] While these are not model systems in the sense of being ultra-clean and having well-defined Hamiltonians, they do offer unprecedented control. This should allow us to test theories of correlated electrons in new ways, for instance seeing directly the effect of changing orbital occupations or dimensionality. Though experimental activity in this area is sharply increasing, few correlated electron theorists have gotten involved. From what I have seen, experimenters are eager for theoretical guidance in how to use these "model" systems, and theorists who believe they have some understanding of exotic physics might have the opportunity to steer some of these structures in that direction.

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Discussion

A. Georges Actually, may I intervene here? Do we know other examples in which the orbital degeneracies are completely lifted in oxides? Somebody has a suggestion? So, cuprates are unique in that respect?

L. Balents Well, for spin 1/2. For higher spins, there is certainly plenty of situations where it is lifted, but spin 1/2 seems very quantum so it is not so easy to find, I do not think.

A. Georges Ok, thank you, so there are a lot of things that have been said here. Let us start with all kinds of issues related to the effect of orbital physics. So, I do not know... Chandra you wanted to say something?

C. Varma It is perhaps a bit foolish to make a cultural remark but I sort of feel that most of the discussion this evening is a cultural matter about which manners of thinking lead to interesting answers in condensed matter physics. And I sort of suspect that there is a lot of ... there is a branch of condensed matter physics which seems to have grown in the last twenty years which I would call *based on particle physics envy*, especially particle physics which has had to do for the last twenty years without experiments. This approach that we will solve a model with not well-controlled approximations and get a very interesting and fascinating result and then go about looking into a small sub-set of experiments on a given material and say "Hey! Have I found it here, have I found it there?" is not the way anything fruitful has ever really happened... I was trying racking my brain to think if I could come up with something interesting that has happened in condensed matter physics that way, and I have not.

A. Georges Frank, you were quoting the Josephson effect?

C. Varma I think the way condensed matter physics traditionally has advanced has been, because it is a very complicated set of interactions in which often something very interesting is hidden, is to look at the variety of things that are happening and to make a hypothesis and then to do some systematic calculation which has some predictive power and then go back and look at the experimentalist and tell them "Gee, does this thing really happen?" and then you are satisfied that something interesting has happened. My guess is that the converse approach which is being followed in this kind of search is not very fruitful.

A. Georges Ok so Chandra is putting us back into discussion mode and waking everybody up. Which is great! Ok, lots of hands are being raised, wonderful ! Let us see... I am going to give the right of speech to an experimentalist. Seamus, you raised your hand...

S. Davis So I was listening for the last several hours trying to understand what would be a good example of a spin-liquid, that an experimentalist could start to do some work on. And here is the question that came into my mind: would a quantum computer without decoherence be a spin-liquid?

A. Georges Surprising! What happened to all the hands? Another experimental-

ist: Phuan Ong...

P. Ong I wish to go back to Leon's remark and bring the discussion back to orbital degrees of freedom. A lot of us have been trying to search for examples of the Kitaev model, and here is where I think orbital degrees may provide a realistic way. This is the suggestion proposed (by Khomskii, Khaliullin and others) which I think is worth following up experimentally. In many transition metal-oxides (I do not remember the actual one proposed by Khomskii and Khaliullin), you can, because of dxy, dyz orthogonality, actually kill the exchange in certain crystalline directions. So by doing solid state chemistry, one can hope to at least realize the Kitaev model in a cubic lattice, and perhaps in the honeycomb lattice as well. (Hidenori) Takagi tells me that they are investigating iridates that are two-dimensional with a honeycomb lattice symmetry. I think bringing the orbital degrees of freedom is going to be a very fruitful direction to go into.

A. Georges I guess I have a question about this: if you are two-dimensional you are typically going to split these orbitals. You can be two-dimensional and keep these orbitals degenerate? Is that what you are saying?

P. Ong You might, yes, with weak 3D coupling. The search goes on, you know, it is not easy to find these systems.

A. Georges Ok, Bernd, you want to comment on that?

B. Keimer Yes, I do not really see a very sharp distinction between what Leon and what Chandra have said actually, and I just want to give you one example which we have been working on and is still ongoing, where all these exact models and/or phenomenological heuristic approaches have gone hand-in-hand, which is perovskites with t_{2g}-orbitals. In titanates we have seen interesting magnetic dynamics, which in turn has inspired theorists to come up with Kugel-Khomskii type models in which the orbitals are treated as iso-spins. So we have a bigger Hilbert space, and you see all sorts of interesting symmetries and perhaps exotic phases such as orbital liquids in the Mott-insulator. And of course these exotic phases are not strictly realized because these orbitals are coupled to the lattice but they have nevertheless inspired us to carry on with our experiments. And the theorists have taken another look at their models, incorporated symmetry breaking terms so I think this is an interesting and inspiring exercise. And there may be ways that Leon has already mentioned to perhaps stabilize these exotic phases using strain from substrates. In this way, one could engineer the lattice distortions to induce the degeneracies that are not easily seen in a material that is not on a substrate. So I think I completely agree that orbital physics is very interesting and the results offer some room for solutions of exact models.

A. Georges So can you quote some classes of materials that you think might be of particular interest?

B. Keimer Yes, these early transition metal-oxides with few d-electrons, like titanates, and I agree that if you go to $4d$ or even $5d$ elements, there are also very interesting systems. So, we worked on ruthanates for instance, which have some near-orbital degeneracy, and we have seen that there are some phases of these

orbitals that are only very weakly coupled to the lattice so these symmetry-breaking terms seem to be much weaker than in other systems.

A. Georges More on the exciting possibilities of orbital physics? Sankar Das Sarma, then Assa...

S. Das Sarma I actually want to make two experimental comments because, you know, I have published a lot of experimental papers actually. So I am an experimentalist for this purpose. First on what Phuan said, I think ... well Alex [Alexei Kitaev] is sitting there I should not really making presumptuous comments on his model but I think the best system to do Kitaev model is perhaps cold atoms because the model is very special: you have many conserved quantities and if the Hamiltonian is slightly different we do not know what is going to happen. So one needs to be careful to just use chemistry in *d*-materials to create the model. I mean it is possible, I am not saying, I have not looked at Khomskii solution deeply enough to say it is not going to happen, but it is a very special model because there are a lot of locally conserved quantities. But I wanted to get back to something ...

P. Ong Sankar, how would you implement that with cold atoms?

S. Das Sarma Well, there are actually several proposals, so I'll discuss it on Monday morning. This is a fairly well developed subject. Leon I was just wondering about something you said, maybe I did not completely appreciate the point: you made the remark that in cuprates we have basically one orbital that... But we have the pnictides where as far as I know we have four or five orbitals and the fluctuations... or something like that. And as far as you can tell they are not that different from cuprates. I mean I am not an expert in it, so the fact that cuprates have one orbital may not be of great significance in their behavior, right? I mean it is a question, I am not challenging you...

L. Balents I suspect if we take a poll in this room, and it would probably be about equally divided, about how important, how similar the pnictides are to the cuprates but that issue is certainly not resolved, I think. Yes, it is very clear that basically single-orbital description is very good in cuprates,

S. Das Sarma But not in the pnictides ? You agree with that?

A. Georges Is very good for what?

L. Balents ... there may be some reason why they have similar low energy physics but certainly in intermediate energy ... that is not a clear statement

S. Das Sarma But do you agree that the pnictides have multi-orbitals ?

L. Balents Sure...

A. Georges Sure, I think that is quite clear.

A. Auerbach Just want to remind the people of *buckyballs*. These are systems which have a lot of local symmetry and therefore their orbital degeneracy in the first excited state is very important both to explain superconductivity in K_3C_{60}, but also interestingly there is a magnetic material which has chains in which the rotation symmetry around the chain is nearly conserved, and then the two orbitals are on the Fermi surface and are important in producing fer-

romagnetism, perhaps even orbital ferromagnetism, it is not clear. But what I want to stress is that orbital degeneracy in C_{60} is both an avenue of getting interesting magnetism *à la* Kugel-Khomskii models, and also of enhancing electron-phonon interaction very strongly beyond what BCS theory will give you because when you solve the local molecular pairing problem you find that the degeneracy actually gives a huge enhancement in electron-phonon coupling. Maybe this is relevant to pnictides.

A. Georges Enhance electron-phonon from orbital degeneracy? More comments on orbital physics?

M.L. Cohen First I should admit that I was a post-doc at the phone company (laughs) and so emotionally I find myself in agreement with Chandra, emotionally. Although I am much more broad-minded than Chandra Regarding these things, I appreciate all his comments. I just want to make an additional comment about C_{60}: in my view, it looks like superconductivity in C_{60} systems is explainable in terms of standard Eliashberg theory . The orbital degeneracy helps with Coulomb interactions since it changes the effective U because you have so many possible channels. But otherwise, I think they have interesting magnetic properties, but I do not think any of the experiments on the systems involving C_{60} are inconsistent with standard BCS theory.

A. Georges May I ask a question on that? Well is there some clear evidence that some of these systems in the absence of doping are actually Mott-insulators?

M.L. Cohen Yes, some of them...

A. Georges So what you are saying is that as soon as we dope them we can forget the correlations?

M.L. Cohen Up to some point, it looks that way...

A. Georges Ok, well,... more along these lines? Well, among the topics I have pointed out at the beginning, there was the question of numerical simulations, and how much can we learn about these 'exotic' states from numerics. We have seen very little of this, up to now, so I guess Matthias may tell us something about that... Matthias Troyer...

Prepared Comment by Matthias Troyer: Quantum Monte Carlo Simulations: Success and Challenges

Strongly correlated quantum many body system provide a wonderful opportunity for finding novel exotic phases but at the same time pose a challenge to our understanding of their properties. Just as experiments can probe for exotic phases in materials, numerical simulations are essential to test whether proposed phases or phase transitions exist in strongly interacting models.

However, these models are also a challenge to simulations since the Hilbert space dimension grows exponentially with system size. Only tiny systems can be solved by brute-force exact diagonalization. The most successful numerical methods for larger

systems are the density matrix renormalization group method (DMRG),[1] reviewed in the article by S.R. White in this volume, and the Monte Carlo method discussed here.

In 1953, Metropolis, Rosenbluth, Rosenbluth, Teller and Teller wrote their seminal paper on the Monte Carlo methods,[2] starting with the famous quote

> *The purpose of this paper is to describe a general method, suitable for fast electronic computing machines, for calculating the properties of any substance.*

Bold as the claim may seem, this is just what the Metropolis *et al.* algorithm can do for classical systems – given a fast enough computer. Over the past 55 years there has been tremendous progress in faster Monte Carlo algorithms, with better convergence properties and speedups of many orders of magnitude,[3,4] but all of these modern methods go back to the original idea of that seminal paper.

While the Metropolis algorithm was devised for classical systems, its application to quantum systems is straightforward after mapping of the partition function of the quantum system to that of an equivalent classical one:

$$Z = \mathrm{Tr}\exp(-\beta H) \equiv \sum_c p_c \tag{1}$$

This maps the operator expression to a simple sum. One standard way of performing this mapping is a path-integral representation in terms of world lines, as originally introduced by Feynman in the same year 1955.[5] The "classical" configurations c here are world lines of the particles in imaginary time, and the weight is given by the corresponding contribution to the path integral. The Monte Carlo updates can be performed by applying the Metropolis algorithm to these world lines.

Besides world lines, any other diagrammatic expansion can be used to map the quantum system to a classical partition function, giving rise to a number of different of representations and algorithms. Over the past fifteen years several efficient quantum Monte Carlo (QMC) algorithms have been developed, including, to name just a few:

- The loop[6] and directed loop algorithm[7] for quantum spin systems allow simulations of millions of unfrustrated quantum spins down to very low temperatures.
- The worm algorithm allows efficient simulations, especially of superfluid properties and Green's functions of arbitrary bosonic lattice[8] or continuum models,[9] with hundreds of thousands of bosons.
- Generalized ensemble methods for quantum systems provide exponential speedup at first order phase transitions.[10]
- Continuous time methods for fermionic systems[11] allow simulations of hundreds of fermions as long as the sign problem (see below) is not too severe.

1. Some Applications

Employing these algorithms a large number of phase diagrams and quantum phase transitions have been explored in the past decades, too numerous to review here. In the context of exotic phases I only want to mention one recent example: the discussion about supersolidity in Helium-4. The supersolid phase of Helium, exhibiting at the same time superfluidity and broken translational symmetry (solid order) has been conjectures forty years ago[12] but only recently has evidence for such a phase been found in experiments.[13] A series of large-scale quantum Monte Carlo simulations[14] for Helium crystals has shown though that the original idea of vacancy condensation in a solid background[12] does not apply since vacancies are gapped and attract and phase-separate in the solid. Instead, crystal defects play a crucial role for supersolidity: superflow happens in defects such as grain boundaries and dislocations. The study of such superfluid crystal defects will be important for a quantitative explanation of the experiments and is a new area of research where ab-initio QMC simulations will be important.

As a second application I want to mention some recent simulations of ultra-cold atomic gases in optical lattices. These "optical lattice emulators" have been widely touted as analog quantum simulators, able to simulate quantum systems where the sign problem prevents QMC simulations. These experiments are now moving from a qualitative to a quantitative stage, and accurate validations with unbiased, fully ab-initio QMC simulations are being done.. Simulating the exact experimental setup with up to 300'000 atoms in traps of 150^3 lattice sites, taking into account effects of finite time of flight in the experiments, finite experimental resolution, heating effects from spontaneous emission from the optical lattice and other relevant details, agreement between experiment and QMC simulation is found for lattice bosons in a cubical lattice. This is the first time a first-principles ab-initio simulation of strongly interacting quantum system is simulated without any approximation and compared to experiments, and is a major stepping stone towards using ultracold atomic gases as reliable quantum simulators.[16]

2. The Negative Sign Problem

While systems with hundreds of thousands of bosons can thus be simulated efficiently and approximation-free, the simulation of fermionic systems suffers from a serious problem: The weights p_c in equation (1) can become negative when fermions are exchanged an odd number of times in a world line configuration. This "sign" problem prevents the direct simulation of the fermionic system since negative weights cannot be interpreted as probabilities. Instead, sampling is done with respect to the absolute weight $|p_c|$ of a configuration and the sign is moves to the

observable to be measured:

$$\langle A\rangle = \frac{1}{Z} = \mathrm{Tr} A \exp(-\beta H) \equiv \frac{\sum_c A_c p_c}{\sum_c p_c} \tag{2}$$

$$= \frac{\sum_c A_c \mathrm{sign_c}|p_c|}{\sum_c \mathrm{sign_c}|p_c|} = \frac{\sum_c A_c \mathrm{sign_c}|p_c|/\sum_c |p_c|}{\sum_c \mathrm{sign_c}|p_c|/\sum_c |p_c|} = \frac{\langle A\rangle_{|p|}}{\langle \mathrm{sign}\rangle_{|p|}} \tag{3}$$

The sign problem now occurs in the fact that the average sign $\langle \mathrm{sign}\rangle_{|p|}$, which is just the ratio of partition functions $Z/Z_{|p|}$ of the fermionic and a corresponding bosonic system, becomes exponentially small, giving rise to exponentially large errors.

The basic physical reason behind the sign problem is the almost obvious fact that we cannot expect to obtain reliable information about a fermionic system (with weights p_c) by simulating a bosonic system (with weights $|p_c|$). As long as the physics of the fermionic system is different than that of the bosonic one, we will just not sample the right phase, nor the relevant configurations!

The sign problem is in general nondeterministically polynomial (NP) hard,[15] and it is conjectured that no polynomial time algorithm exists for such problems. Still, in special cases, like fermions with attractive contact interaction or in half-filled bands with repulsive interactions, the sign problem can be avoided. In all of these cases the fermionic nature of the state does not play a role and a "bosonic" simulation gives reliable results: a BEC state for attractive interactions and a Mott-insulator for repulsive ones. In general Fermi-liquid or non-Fermi-liquid phases however, no suitable "bosonic" picture that could be used as a basis for QMC simulations has been found yet. This is the biggest remaining challenge of achieving Metropolis *et al.*'s goal of "calculating the properties of any substance".

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Discussion

A. Georges Ok, maybe you can give us some feeling of what can or cannot be solved... in a nutshell.

M. Troyer In a nutshell we can solve on big 2D lattices anything which is essentially classical, that has some broken symmetry states, that is superfluid, thus bosons... Fermions are hard. Because I think they are intrinsically in some way non-classical.

A. Georges Ok, thank you, that is a clear-cut answer. Bert...

B. Halperin You mention the possibility of supersolids being explained by superfluidity around defects. Do you have some estimate of what the superfluid density would be, or how many defects you would need in order to explain the numbers that come out of the experiments?

M. Troyer In the past, they have seen superfluid density around 0.1 %, that is a number which one can explain with the density that is there. There are numbers that go up to 20-40 % , I would think that these samples are probably not solids. There might be some glasses, or mixtures of liquid bubbles and some solid, so the current data cannot all be explained by superflow. Some DC flow data are very slow and that flow can be explained through defects.

A. Georges So I have seen the hands of two famous experimentalists rising: Seamus and... Chandra Varma !

S. Davis Just a comment about the superfluid density. Those experiments do not measure the superfluid density. They measure the mass density which is decoupled from the moving apparatus. So when you look at the numbers they report, the superfluid density could be enormously higher than what they report, it is a function of the network through which the liquid is flowing and that network is unknown.

C. Varma I have a question on numerical experiments, which I think are wonderful because they can rule out certain things. And so my question is: there is this work by Imada saying that to a large extent he has solved the fermion sign problem in quantum Monte-Carlo and concluded that the single-orbital model, the Hubbard model, for a range of parameters, if it has superconductivity at all, it is below some unbelievably small number. I just wondered what the status of quantum Monte Carlo is, with respect to the fermion sign problem, is in that regard, and is this result generally supported by independent calculations?

M. Troyer Yes, I wanted to mention that on Monday but I can mention it now.

A. Georges We will discuss it more on Monday... so maybe in a nutshell for non-

specialists you can summarize the point?

M. Troyer In a nutshell, that is a method that avoids the sign problem by replacing it by a similar model but for weakly interacting Fermi systems, so Hubbard model with a diminished U. On small system sizes it seems to work. So the results there are reliable in the sense that it is a projection method. So he starts projecting from the Hartree-Fock state which has weak-pair-correlations only. And if that is a good trial wave function, then the results which you get are reliable. If the true ground state is very close in energy but has a very different structure, then we will might still be fooled. So that is the only thing that might happen there.

A. Georges So, the punch line of this work which is that he did not observe d-wave superconductivity, or strong correlations, do you think this is still subject to the caveats that you have just mentioned?

M. Troyer It is subject to the caveat that he starts from the trial wave function. The method should be unbiased, the question is: does he project far enough ? He tried starting from Hartree-Fock wave function, and did not see increasing the correlations as he projected. He tried starting from a d-wave BCS wave function, and did not see any difference either. So, if one finds a new wave function that one could try with, maybe it looks different, but so far that is the state-of-the-art.

A. Georges So we are running a bit out of time here, the session has gone overboard, we shift now from fermions to bosons and even other statistics. Does anybody want to say something more on bosons? Assa, you want to say a few words?

A. Auerbach I just wanted to mention that bosons also pose some interesting challenges. People always think that bosons have been solved a long time ago, but once you put bosons on lattices, we get very interesting phases. We know the Mott phases, and the superfluid-Mott transitions have been seen in optical lattices. I think also they might have relevance to the phase diagrams of cuprates, if you look at the zero temperature superfluid density of hard-core lattice bosons ... But the interesting things that we find, and actually leaves more questions open, are the transport properties of bosons on lattices. It turns out that the Hall effect for instance, which could be measured by the Chern number, exhibits very interesting structures. Actually at weak interactions, the Hall coefficient is simply a measure of the total number of bosons, but when you go on to optical lattices with strong interactions, you get that the Hall coefficient vanishes at integer fillings, and then it also vanishes at half-integer fillings, and it changes its sign dramatically, and associated with it we find behaviors that look like quantum critical points related to the original title of the session. But the Hall coefficient at half filling changes abruptly with seemingly vanishing energy scale and exactly at half filling, if you look at the states – the actual many-body eigenstates of hardcore bosons at half-filling on the lattice – all the states are two-fold degenerate, they seem to acquire spin 1/2. It looks

like in a sense, a new kind of quasiparticles emerge at that point, so it looks like a quantum critical point observed in the transport behavior of bosons on lattices. That is what I wanted to say.

A. Georges Thank you, Assa.

A. Stern Is this is all numerical, Assa ?

A. Auerbach No, the proof of the spin-1/2 properties and the degeneracies is done by finding non-commuting operators at half-filling, that have an $SU(2)$ algebra, and therefore all the states are doubly degenerate. That holds for any finite lattice. The Hall effect jump was done numerically at different lattice sizes and they all exhibit the same kind of jump or the same vanishing energy scale. But that was finite 4×4, 4×5, lattices.

A. Georges Ok, some comments? More comments on bosons? Senthil you wanted to say something?

T. Senthil Yes, so listening to Chandra and listening to all these talks, I wanted to say some things. First is that I love spin-liquids, worked on it for a long time, but I think that in this field of correlated electrons it is useful to remind ourselves of some of the motivations for talking about spin-liquid physics. One of the motivations really comes from the fact that there is no dearth of exotic phenomena if we are willing to look at metallic systems. There is this whole set of exotic experiments for the last twenty years, and it is really embarrassing for theory in this field that we are more or less completely useless to experiments. So, part of the reason for studying spin-liquids in insulating quantum magnets is as theoretical investment for learning about the kinds of things that correlated electrons can do in a simple context and hopefully eventually one will be able to think about metals. But in terms of going towards metals I think already what is being found theoretically in spin-liquid physics, the results that are coming out, are pretty amazing: we have been able to describe phases which do not have any quasiparticles, description of critical points that violate Landau paradigms and so on... So can we tentatively start making a move towards describing exotic metals? I want to put up one question which I think is – from recent experiments you realize it is – quite fundamental to the field and that is really something where the fact that it is a metal comes in crucially. That is the question of how a Fermi surface might die. So it turns out that in many of circumstances in which you empirically get exotic physics, which is in metals, many of them are close to various kinds of quantum phase transitions, and it appears as though these are phase transitions in which an entire Fermi surface disappears. The best studied example perhaps is the onset of magnetism in heavy electron materials. So, in the heavy Fermi liquid phase, as many of you know, there is some Kondo screening which absorbs the local moments that are there in those materials into the Fermi sea, and expands the volume of the Fermi surface. The other phase in question, is an antiferromagnetic metal phase which has a very different Fermi surface, apparently, from experiments, and it appears that as though there is a direct phase transition between these two

phases. At the critical point – which has been studied to death in experiments – there is a striking non-Fermi liquid and absolutely no theory that is worth talking about at this critical point.

Similar issues of course arise in a much simpler theoretical system. If you think about the Mott transition of the single band Hubbard model, at half filling, it goes from a Fermi liquid (large t/U) to an antiferromagnetic Mott insulator (small t/U). The Fermi liquid has a Fermi surface but no magnetism, the Mott insulator has magnetism but no Fermi surface. Could there be a direct phase transition between these two limiting values ? And that question is clearly closely tied to the phenomena that has already been observed in the heavy electrons systems. And finally, as Subir mentioned, there is the high-T_c materials with the possibility of a direct transition from overdoped to underdoped.

In all these cases, if the transitions are second order, and in the heavy electron systems it appears to be second order, the critical point will be non-Fermi liquid, and we have to understand that kind of critical point perhaps based on whatever has been learned theoretically on spin-liquid physics.

A. Georges Well, ok, so actually this connects very nicely to tomorrow's session as well, right? So we can raise the question and continue that tomorrow...

T. Senthil I think it does. So let me just put up two questions that I hope will be answered in the future in this field... The first is: can the Fermi surface disappear at the same point as the magnetic phase transition? Now, empirically for the heavy fermions, that seems to happen. So we do not know the answer to this question but here is an example where spin-liquid physics has been useful. Similar kinds of questions – but much simpler versions – can be posed and answered in the context of insulating quantum magnets. So I think we are learning something, even in the traditional culture for the experimental problems in the field. Second, the question of how are we going to think about phase transitions where an entire Fermi surface disappears, right? Again, for heavy fermions that is certainly an issue, and one thing that we might imagine is that the critical modes at the phase transition live on the entire Fermi surface. So the Fermi surface at some level has gone critical... right? There are various arguments that one can make that at such a phase transition, we loose the Landau quasiparticles, but the Fermi surface retains its sharpness, so the kind of phenomenon that we have to deal with is one in which the critical modes live on an entire surface in momentum space. Which means for instance the scaling phenomenology is going to be completely different from usual criticality. So this kind of thing has enormous potential for describing various kinds of bizarre phenomena that have been reported in the field for twenty years and for which we basically do not have any reasonable starting point to understand. But of course a longer term question is: what calculation framework can be developed to deal with such problems where entire Fermi surfaces disappear?... That is all I wanted to say!

Prepared Comment by Todadri Senthil: Killing the Fermi Surface: Towards a Theory of Non-Fermi Liquid Metals

In the last two decades a number of remarkable experiments on diverse materials have demonstrated the failure of Landau's theory of Fermi liquids in some correlated metals. The most striking example perhaps is the 'strange metal' that occurs in the cuprate high temperature superconductors near optimal doping. Another striking example is provided by heavy electron metals such as $CeCu_{6-x}Au_x, YbRh_2Si_2, ...$ in the vicinity of a magnetic quantum phase transition. Several others are being routinely unearthed. Despite this growing empirical observation there is very little theoretical understanding of non-fermi liquid physics in any system. An important clue may possibly be found in experiments done in the last few years. It appears that the non-fermi liquid physics may be closely tied to a kind of (quantum) phase transition phenomenon where an entire Fermi surface disappears. This has been discussed most extensively in the context of the heavy electron non-fermi liquids that develop near the onset of magnetism in a heavy fermi liquid (for reviews see Refs. 1). The heavy fermion system is well modelled as a 'Kondo lattice' i.e a lattice of local moments (formed by f-electrons) coupled through Kondo exchange with a separate band of itinerant conduction electrons. In the heavy fermi liquid phase, the local moments are absorbed into Fermi sea through the process of Kondo screening. The resulting 'large' Fermi surface satisfies Luttinger's theorem on the Fermi surface volume only if the local moments are included in the count of the electron density. In the antiferromagnetic metal on the other hand it is possible that the local moments freeze due to RKKY exchange interactions before any Kondo screening process can set in. It is natural that the local moments are not part of the Fermi sea in such a metal. Hence it is possible that the Fermi surface in the magnetic side has a rather different shape and size (loosely put, a 'small' Fermi surface) from the large Fermi surface of the paramagnetic heavy Fermi liquid. Thus a direct phase transition between these two phases requires (in addition to the magnetic ordering) a dramatic change of the electronic structure associated with the death of the large Fermi surface and its replacement by a small Fermi surface. Remarkably recent experiments have provided evidence[2,3] for such a dramatic Fermi surface change at these heavy fermion phase transitions. Furthermore the transition itself seems second order. Thus it appears likely that entire sheets of the Fermi surface disappear *continuously* at this quantum phase transition. A similar phenomenon also seems likely to be happening in the 'underlying' normal ground state of the hole-doped cuprate high temperature superconductors. In the overdoped side a number of experiments have convincingly established that the underlying normal ground state is metallic with a 'large' Fermi surface[4] whose area is set by the total electron density ($1-x$/unit cell if x is the hole density). In the underdoped side it seems that the underlying normal ground state is metallic. There is considerable confusion on what the Fermi surface of this state actually looks like (or even if it has a Fermi

surface in the conventional sense at all). Nevertheless it seems certain that the Fermi surface is dramatically different from the large Fermi surface of the overdoped side. Rather it seems likely to have small pockets[5] with area close to x rather than $1-x$. Thus with decreasing x the large Fermi surface needs to disappear through a phase transition. It is tempting to make an analogy with the heavy fermion example above, and explore implications for strange metal physics above the superconducting transition.[6] A simpler example also illustrates the possibility of the death of an entire Fermi surface. Consider a two or three dimensional system modeled as a single band Hubbard model at half-filling on a non-bipartite lattice. If the hopping matrix element t of this model dominates the on-site repulsion U a stable Fermi liquid with a Fermi surface satisfying Luttinger theorem results. If on the other hand U dominates over t, a Mott insulator with no Fermi surface results. If the Mott transition is continuous then the entire Fermi surface of the metal needs to disappear continuously. Again recent experiments raise the tantalizing possibility of such a second order Mott transition.[7] The possibility that an entire Fermi surface might disappear through a continuous second order transition is an intriguing one. It certainly has the potential to underlie the strange metal physics observed in many of these systems. Clearly when the Fermi surface is on the verge of disappearing it is very natural to expect Fermi liquid theory to break down. In many of the examples above the disappearance of the Fermi surface is also accompanied by appearance of other phenomena (such as magnetism for instance). Can the Fermi surface disappear at the same point as the magnetism appears as the experiments apparently suggest? The answer is not known. The problem is that two seemingly separate phenomena (change of Fermi surface and magnetic ordering) need to occur at the same value of the tuning parameter that drives the transition. Interestingly similar questions arise in the much simpler context of insulating quantum magnets where they can be answered.[8] A number of examples have been described of direct second order phase transitions between two phases which either break distinct seemingly unrelated symmetries or between a broken symmetry phase and a spin liquid phase with 'topological' order. The resulting theory – dubbed 'deconfined quantum criticality' – is naturally formulated in terms of unusual 'fractionalized' variables rather than the more usual Landau order parameter. Perhaps these simpler examples will show the way forward in the metallic case as well. A more fundamental question is how in the first place an entire Fermi surface might disappear continuously? One idea goes back to early work on the Mott transition.[9] If the quasiparticle residue Z vanishes continuously everywhere on the Fermi surface as the transition is approached we can lose the entire Fermi surface in one shot. (Concrete examples of this kind of transition in two or three dimensions can be studied with 'slave particle' techniques and provide some insight.[10–12]) A crucial next question is the fate of the Fermi surface right at the quantum critical point when Z has just gone to zero. Recently it was argued[6] that such a quantum critical state will continue to have a sharp Fermi surface even though (as Z is zero) the Landau quasiparticle is gone. This was dubbed a ' critical Fermi surface'. Clearly the presence of a critical Fermi surface will

severely alter the phenomenology when compared with any conventional quantum critical point. Scaling hypotheses for such a quantum critical point were proposed in Ref. 6. Developing a calculational framework to describe critical Fermi surfaces is an important challenge for the future.

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Discussion

A. Georges Natan wants to say something about this...

N. Andrei Maybe I have a more general question: suppose you could do a dynamical mean-field theory, how would all these exotic phases manifest themselves. Or could you use, if you understand how it manifests itself in an effective spin model, effective impurity model, can you project backwards, and set up a model that would have these properties ?

A. Georges Ok, that is going to be a discussion between the chairman and Natan Andrei... What about a private discussion on this? I would be happy to elaborate on that...

N. Andrei Ok...

A. Georges I think we have basically to close and I am sorry for people who actually prepared something and wont have time to present it. Nevertheless, before we do that, there are two issues that were on the list that we have not have time to say anything about. One is interesting dynamics, and I was just wondering whether Leticia wants to say, maybe in the microphone, just a few words about this...

L. F. Cugliandolo As many of you know, I have been interested in aspects of glassy dynamics, and glassy dynamics was mentioned a little bit this morning. So, basically, what one can do is to solve simple models, very simplified models which are in a sense on the other end in comparison to the ones that Subir Sachdev have discussed today. In the sense that they are defined on random graphs: very large dimensional models if you wish, and those can be solved and they show a number of peculiar features at the classical level, but also if you switch on quantum fluctuations. And just very briefly (if you want we can discuss it privately later on) the features I wanted to mention were that quantum phase transitions very typically become first order, so this is something that has to be taken into account. Even if the classical one is second-order, the quantum one close to zero temperature might become first-order. Then the dynamics in the ordered phase is of course very slow, it has features of glassiness with separation of time scales, and all these sorts of strange behavior. And also that the effect of the environment is very important and that you can even change the location of the phase transition by changing the coupling to the environment. So this is something, in a sense, surprising. Some other thing that can be done is to drive the system out of equilibrium still more by passing a current through it and looking at what happens along this new extra axis that you are including by this driving strength.

Prepared Comment by Leticia Cugliandolo: Dissipative Quantum out of Equilibrium Dynamics

In recent years quantum out of equilibrium phenomena have grown in importance. The current ability to prepare cold atoms in good isolation from the environment and to tune some Hamiltonian parameter such as the interaction strength (quantum quench) is opening the way to the experimental study of out of equilibrium relaxation in closed systems.[1] The interest in this situation is also boosted by the potential use of quantum annealing procedures to get close to optimal states in hard problems. On the other hand, the coupling of a system to an environment cannot always be avoided and it may provide a severe source of dissipation and de-coherence. Indeed, the experimental realization of quantum simulators and quantum computers is limited even by very weak dissipation. Further, coupling a system to an environment can lead to dramatic effects such as dissipative quantum phase transitions.[2] Possibly the most studied realizations of quantum dissipative systems are nanoscopic ones due to their technological relevance, see *e.g.* the series of lecture notes in[3] for a review. But also interesting are larger cases in which a parameter change can set the system out of equilibrium and generate a host of new and challenging phenomena. In particular, glassy dynamics, that is to say the impossibility to reach equilibrium with the environment in laboratory time-scales, has been found in large dissipative systems at very low-temperatures, where thermal

fluctuations are negligible but quantum fluctuations play a rôle. Examples showing these features are the dipolar magnet[4] $LiHo_xY_{1-x}F_4$ and the high-T_c compound[5] $La_{1-x}Sr_2Cu_2O_4$.

The theoretical understanding of these phenomena is not satisfactory yet. In the context of quantum quenches one would like to have a general comprehension of the asymptotic state; under which conditions a subsystem reaches a state described by a canonical density matrix; *etc.* Numerous groups around the World are currently studying these questions and related ones.

In the case of dissipative quantum non-equilibrium relaxation one would like to, at least, attain the same level of understanding existent in the classical limit and, in so doing, identify the genuinely quantum features. In this context, issues that are currently under investigation for quantum systems were already addressed in the classical case. Indeed, the relaxation after an instantaneous quench – typically realized as an abrupt change in the temperature of the bath taking the system from the disordered to the ordered phase – has been studied in great detail especially in systems evolving through the growth of domains of two competing equilibrium states ('coarsening'), the typical example being clean and dirty ferromagnetic materials; and generic glassy systems for which the underlying dynamic mechanism is still not known, such as window glass or polymer melts. None of these systems reaches a steady state in laboratory time-scales and several aspects of their relaxation are very similar including the so-called aging effects – or the fact that older systems have a slower relaxation. The level of understanding of coarsening-like non-equilibrium phenomena is quite good: the phenomenological dynamic scaling hypothesis whereby there is a single growing length-scale characterising the dynamic evolution yields a very satisfactory description of numerical and experimental data.[6] Instead, the comprehension of glassy dynamics is only partial. Much theoretical progress has been achieved by solving analytically the dynamics of mean-field models[7] but an understanding of the behaviour of finite dimensional cases is far from complete and several competing viewpoints are pushed by different groups. On a slightly different front, the study of driven – not necessarily glassy – dissipative classical systems has also lead to important new notions such as out of equilibrium phase transitions[9] and fluctuation theorems out of equilibrium.[10]

The necessity to develop powerful tools to treat the quantum extension of the problems exposed in the previous paragraph and derive a – as complete as possible – scenario for quantum out of equilibrium dynamics is becoming compelling. An important effort has been set on the study of disordered mean-field models and their out of equilibrium relaxation. As techniques are concerned, most of the analytic methods to study equilibrium (replica field theory and the cavity method), metastability (the Thouless-Anderson-Palmer approach) and real-time dynamics (Schwinger-Keldysh combined with a path-integral treatment of the model environments) have been developed. Some interesting results obtained are the following.

First-order quantum phase transitions are the rule rather than the exception in disordered models related to hard optimisation problems of the random K-Sat

type[11] (models with random multi-uplet interactions between spins). The fact that the potential efficacy of quantum annealing procedures will be severely altered in problems with first-order phase transitions since the gap between lowest lying states is exponentially small in the system size was recently discussed.[12] Interestingly enough, a first-order phase transition between a paramagnetic and a glassy phase was also found experimentally[4] in the transverse field dipolar-coupled Ising magnet $LiHo_xY_{1-x}F_4$ at intermediate dipole concentration x.

The static and dynamic phase diagrams depend on the coupling to the environment and on the type of environment used.[8,13,18] For instance, the ordered ferromagnetic phase of a system of spins that are independently coupled to ensembles of Ohmic quantum oscillators or electronic leads are enhanced by a stronger coupling to the environment. An intriguing consequence is that a system with identical Hamiltonian parameters can behave ferromagnetically or paramagnetically depending on the coupling the environment.

After a sudden change of the environment parameters (temperature, coupling constant, *etc.* such that the conditions are the ones of the ordered phase macroscopic quantum systems with ferromagnetic interactions undergo coarsening[18] and more complex glassy models also age[14] although, as in the classical limit, there is no clear comprehension of which are the real-space processes taking place.

The notion of an effective temperature characterising the out of equilibrium slow relaxation of classical glassy systems[15] has helped interpreting several aspects of their dynamics in intuitive terms. It was initially defined using the deviation from the dynamic fluctuation-dissipation theorem and its thermodynamic properties were later checked. The slow relaxation of quantum mean-field dissipative glassy models is also characterised by deviations from the quantum fluctuations-dissipation theorem that actually takes a classical form with the bath temperature replaced by a different value, the effective temperature.[14] This feature has been interpreted as a (time-dependent) de-coherence phenomenon whereby the large-scale relaxation looses the quantum information and, for all purposes becomes classical with 'renormalized' parameters. The thermodynamic meaning of the effective temperature in such quantum glassy systems has not been explored in sufficient detail yet.

Recent studies of driven, *e.g.* by a current, quantum systems demonstrate that disorder-order phase transitions can survive the effect of the drive until a critical voltage value is reached.[16,18] Interesting features of the zero-temperature critical line, that bear some resemblance with jamming transitions of classical athermal systems,[17] have been addressed.

In conclusion, understanding the out of equilibrium dynamics of isolated and dissipative quantum systems is one of the most challenging open problems in physics. Such a project is of clear fundamental relevance but it also has practical importance pushed by the current activity in cold atomic gases and the possible technological applications of nano-devices. We anticipate that much progress will be achieved in the near future in all the involved fronts: theoretical, experimental and technological.

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Discussion

A. Georges More about out-equilibrium physics of quantum systems will probably be discussed in the cold atoms session on monday. We heard a lot about the Kitaev model in this session, so it makes sense to give Alexei time in a proper way tomorrow. Concluding this session: this is a very active field!

Session 3

Experimentally Realized Correlated-Electron Materials

Chair: *Maurice Rice*, ETH Zürich, Switzerland
Rapporteur: *Catherine Kallin*, McMaster University, Canada
Scientific secretaries: *Nathan Goldman* (Université Libre de Bruxelles) and *Michele Sferrazza* (Université Libre de Bruxelles)

Rapporteur talk: Experimentally Realized Correlated Electron Materials: From Superconductors to Topological Insulators

1. Abstract

Recent discoveries, as well as open questions, in experimentally realized correlated electron materials are reviewed. In particular, high temperature superconductivity in the cuprates and in the recently discovered iron pnictides, possible chiral p-wave superconductivity in strontium ruthenate, the search for quantum spin liquid behavior in real materials, and new experimental discoveries in topological insulators are discussed.

2. Introduction

Nature has provided us with an incredibly diverse variety of materials which exhibit striking phenomena driven by electron correlations. Partially driven by the discovery of new materials, there has been significant recent progress in understanding correlated electron systems which do not fit into our wide-reaching paradigms of Fermi liquid theory and spontaneous symmetry breaking order, although many

challenging open questions remain. It is not possible to review all the interesting strongly correlated materials which are currently under active investigation, so I will primarily focus on some of the newer superconductors, as these are not covered elsewhere in the Proceedings and these especially have stimulated enormous scientific effort, leading to many new ideas. In particular, I will focus on the high temperature cuprate superconductors, the iron pnictides, and possible chiral p-wave superconductivity in strontium ruthenate. Newly discovered topological insulators will also be discussed. A number of related experimental systems are discussed elsewhere in the Proceedings. In particular, frustrated quantum magnets (mentioned briefly below) are treated in depth by Sachdev,[1] and quantum Hall systems are the subject of the article by Stern.[2]

3. Unconventional Superconductors

Unconventional superconductors are those in which superconductivity arises from direct electron-electron interactions, as contrasted to the conventional indirect interaction via phonons. Direct interactions often favor higher (than s-wave) angular momentum pairing. Although the normal state of the high temperature superconducting cuprates is not a conventional Fermi liquid, so the concept of pairing electron-like quasiparticles may not be completely valid, it is known that the on-site Coulomb repulsion and spin fluctuations play a key role in stabilizing the d-wave superconducting state. A completely new class of high temperature superconductors, the iron pnictides, were discovered just in the last year and their pairing symmetry is still under investigation, as is the question of whether the mechanism for superconductivity in these Fe-based materials is closely connected to that of the cuprates or whether a new route to high temperature superconductivity has been found. Another novel superconductor which has attracted considerable recent attention is strontium ruthenate, Sr_2RuO_4. Ferromagnetic spin fluctuations are believed to be responsible for the superconductivity in this material, but the interest here is not due to a high transition temperature (in fact, T_c is only 1.5K) but because experiments point to a chiral p-wave order, which is a topological order that can, under certain conditions, support quasiparticles with non-Abelian statistics. Intense effort in understanding each of these novel superconductors has led to many new ideas and new paradigms about the type of behavior quantum many-body systems can exhibit. Specific highlights in our current understanding as well as open questions surrounding each of these superconductors are reviewed below.

3.1. *High temperature cuprate superconductors*

Over the last two decades, the superconducting cuprates have been the most intensely studied materials in physics. Much of this interest stems from the high superconducting transition temperatures, T_c, and the consequent potential for new applications. Whereas the maximum observed T_c had slowly increased from 4.2K in 1911 (in Hg) to 23K in 1974 (in Nb_3Ge), following the discovery in 1986 of su-

perconductivity at 35K in $La_{2-x}Ba_xCuO_4$,[3] the highest T_c quickly shot up to 138K (or higher under pressure) as many other cuprate oxides were discovered.[4]

Intense interest in the cuprates also follows from the strong role that electron-electron interactions play in these materials. Although the correct and complete theory of high temperature superconductivity is still under debate, much is now understood about the behavior of these materials. More generally, attempts to understand strong electronic correlations in the cuprates have generated many new ideas, particularly in the area of quantum magnetism, as discussed by Sachdev in these Proceedings.[1] Research in the cuprates has led to a much deeper understanding of non-Fermi liquid behavior, particularly quantum order or topological order.[5]

Many different materials belong to the class of cuprate superconductors. A few well-studied examples are $La_{2-x}Sr_xCuO_4$, $YBa_2Cu_3O_{6+x}$ and $BiSr_2CaCu_2O_{6+x}$. What the cuprates all have in common is fairly weakly coupled copper oxide layers (called planes) which are where all the electronic action is. The material between these planes acts as a charge reservoir, and changing the crystal stoichiometry (*i.e.* changing x in the chemical formula) changes the electron density, or the "doping", p, of the copper oxide layers. This leads to a temperature versus doping phase diagram, as shown in Fig. 1.

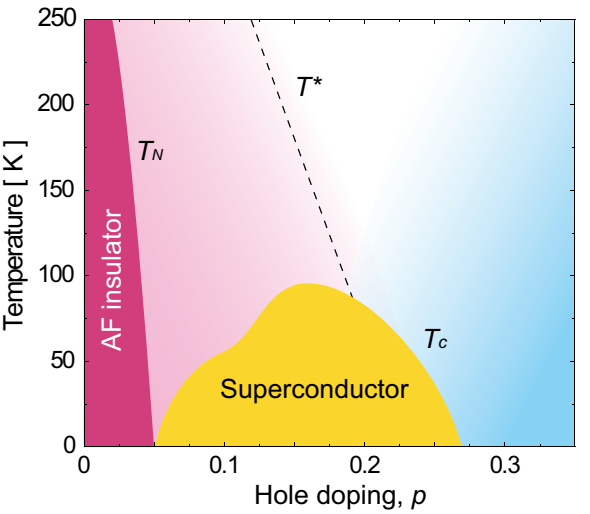


Fig. 1. Cuprate phase diagram as a function of hole doping. The doping where the maximum in T_c is achieved is referred to as optimal doping. The pseudogap phase appears below the crossover temperature, T*. From Ref. 23.

The undoped phase corresponds to exactly one electron per Cu site, which band theory would predict to be a metal. However, the undoped phase is a Mott insulator, due to electron-electron interactions. A strong on-site Coulomb potential, U, localizes the electrons, one to each Cu site. The electronic correlations also lead to

antiferromagnetic order in this phase. With increasing doping, the material becomes a superconductor. The pairing order parameter is known to have d-wave symmetry, with nodes in the superconducting gap along the directions $k_x = \pm k_y$.[6,7] While weak-coupling BCS theory for a d-wave order parameter can be a useful starting point for describing the superconducting phase at low temperatures, there are important deviations from BCS theory. In particular, in the underdoped region (doping less than the optimal doping where the maximum T_c is achieved), the superfluid density and T_c fall off with decreasing doping, whereas the superconducting gap increases.[8] This behavior is believed to result from strong correlation physics and to be a signature of a doped Mott insulator. Indeed, it follows quite naturally from strong correlation theories inspired by Anderson's resonating valence bond picture.[9] At sufficiently large doping, the normal state appears to be a more or less conventional Fermi liquid, whereas in the underdoped and optimally doped region, the normal state is anomalous. Shown in Fig. 1 is a cross-over temperature, T*. The anomalous state below T* is called the pseudogap phase, since the low-energy density of states and the spin susceptibility are suppressed in this phase. There is no observed phase transition at T*, but most physical properties undergo a smooth but substantial change at this cross-over temperature.

The key theoretical goal underlying research in high temperature superconductivity is to explain all universal properties in the insulating, pseudogap and superconducting phases within a theory which can make verifiable predictions. Much of the current attention is focussed on the pseudogap region for this purpose. Part of the reason is that, while the ground states of the Mott insulating and the d-wave superconducting phases are understood, the nature of the pseudogap ground state, or whether it is even connected to a ground state as opposed to being a strongly fluctuating phase associated with the insulating and superconducting phases nearby, is still a point of debate.[10] Furthermore, the pseudogap phase is generally viewed as the key to understanding the cuprates since it occupies a large region of the phase diagram in temperature and doping, it connects the strongly correlated Mott insulating phase to the high temperature superconducting phase, and, most importantly, it is the normal phase from which the superconductor condenses over much of the superconducting dome. Understanding the pseudogap phase is seen as equivalent to understanding the doped Mott insulator.

There are many different ideas and proposals for the pseudogap phase, including preformed Cooper pairs,[11] antiferromagnetic and/or superconducting fluctuations,[12,13] static and fluctuating stripes or nematic order,[14] staggered flux[15] and d-density wave[16] phases, and orbital currents.[17] The staggered flux phase emerges from the resonating valence bond (RVB) picture, which captures much of the cuprate phenomenology.[18] Part of the difficulty in understanding the pseudogap phase is that experiments see evidence for many of these different behaviors, at least in some materials in some parts of the pseudogap region, and it is then a question of which one, if any, is key to high temperature superconductivity. Below, in these proceedings, Varma[19] makes the case for the importance of orbital currents,

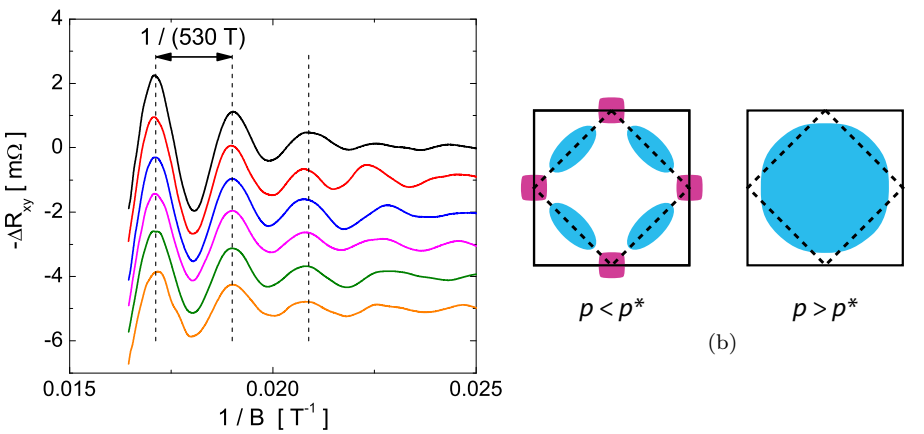
as several experiments have seen evidence for this order in the pseudogap phase.[20] Kivelson and others[14] have argued that fluctuating stripes may be central to high T_c superconductivity. Fluctuating stripes, as well as the preformed pairs proposal, might be considered precursor theories, in that the proposed order is a precursor to obtaining high temperature superconductivity. Other proposals can be classified as competing orders, order which competes with superconductivity and leads to the distinctive phase diagram observed. D-density waves[16] are an example of competing order, as are static stripes. Detailed studies of the phase diagram may distinguish between precursor and competing theories as one would expect the crossover temperature, T*, to slice through the superconducting dome, presumably ending in a quantum critical point at T=0 under the superconducting dome in the case of competing order. By contrast, one would expect T* to hug the superconducting dome, merging together with T_c on the overdoped side if the pseudogap is a precursor effect. In fact, both types of behaviour have been seen in experiment, depending on which physical property or signature one tracks at T*, suggesting that both precursor and competing signatures are present in the pseudogap phase.[21] In addition to T*, there is a lower cross-over temperature below which one observes an unusual Nernst signal, which is interpreted as evidence for superconducting pairing without long-range phase coherence.[22]

Here, I will focus on one particular set of experiments which address the nature of the pseudogap phase and which have generated enormous interest – recent observations of quantum oscillations in the pseudogap region.[23] First, let me briefly review the relevant ARPES results. In the overdoped regime, a single large Fermi surface, centered at (π, π) and enclosing 1+p holes per Cu site, where p is the hole doping, is observed.[24] This is exactly what one expects from band theory. Something quite different is observed in the underdoped regime. ARPES shows four "Fermi arcs" centered at the nodal points near $(\pm\pi/2, \pm\pi/2)$.[25] How does one explain the observation of pieces of Fermi surface which are neither closed orbits nor open orbits intersecting the Brillouin zone boundaries? One possibility is that there are small hole pockets centered at the nodal points, but due to matrix element effects, only one side of each pocket is visible in the experiments. Alternatively, there are strong correlation theories which can account for such arcs.[26] Furthermore, the observed arcs are temperature dependent and, at least in some cases, it has been shown that the arcs extrapolate to nodal points at zero temperature.[27] All of these scenarios, Fermi arcs, Fermi nodal points, or small hole pockets in the absence of any long range order which breaks a symmetry, are incompatible with Fermi liquid theory and Luttinger's theorem and do not connect smoothly to the large Fermi surface observed at larger dopings. (Luttinger's theorem says that the area enclosed by the Fermi surface is the same as for non-interacting electrons.) Furthermore, in the underdoped regime, the superfluid density scales with hole doping (despite band theory predicting a less than half filled electron band) which suggests that this regime is more closely connected to the Mott insulating antiferromagnetic phase at zero doping than it is to the metallic phase of the overdoped regime. These and

other results have led most groups to focus on non-Fermi-liquid descriptions of the pseudogap phase of the cuprates.

Therefore, it came as a surprise when Proust, Taillefer and coworkers[23] observed quantum oscillations in the longitudinal and Hall resistance of underdoped YBCO, apparently establishing the existence of a well-defined Fermi surface when the superconductivity is suppressed by a magnetic field. The cross-sectional area of the Fermi surface can be extracted from the period of these oscillations. Experimental data for the Hall resistivity, exhibiting three clear periods, is shown in Fig. 2(a). More recent data shows up to eight periods, leaving little doubt that the period is proportional to the inverse magnetic field.[28] The Fermi surface area extracted from these data is tiny, about 30 times smaller than the Fermi surface area observed in the overdoped regime, and too small to be consistent with hole pockets centered at the nodal points and enclosing p holes per Cu, where p is the hole doping of the sample. Furthermore, the negative Hall coefficient at low temperatures at this doping of YBCO is taken as evidence that the carriers are electrons, not holes.[29] However, small electron pockets are incompatible with Luttinger's theorem. This led Taillefer and coworkers to propose a Fermi surface reconstruction, leading to hole pockets near the nodal points and electron pockets near the zone boundaries as shown in Fig. 2(b). This proposal is then compatible with Luttinger's theorem, but raises several other questions.

First, this type of Fermi surface reconstruction is what one would expect in the



(a) (b)

Fig. 2. Quantum oscillations and Fermi surface, taken from Ref. 23. (a) Oscillatory part of the Hall resistance of underdoped YBCO as a function of inverse field, 1/B, at temperatures ranging from 1.5K (top curve) to 4.2K (bottom curve). (b) Reconstructed Fermi surface proposed to explain quantum oscillation data in the underdoped (pseudogap) region and the large Fermi surface observed in the overdoped region.

presence of charge or spin density wave order which introduces a new periodicity. Possibilities include antiferromagnetism, d-density wave order,[16] or stripe order.[30] However, no such long-range order has been observed in the pseudogap phase. It has been suggested that sufficiently slow fluctuations in any of these orders might account for the proposed reconstruction, although this suggestion remains to be quantified and compared in detail to quantum oscillation and other experiments. A second point which needs to be addressed is the absence of any signature of hole pockets in the quantum oscillation (SdH and dHvA) measurements, although this could be explained if, because of higher mobility, the electron pockets dominate the signal at low temperatures. Finally, there is the question of how to reconcile the picture of a reconstructed Fermi surface with ARPES data which sees only Fermi arcs near the nodal points. In particular, the quantum oscillation data appear inconsistent with the proposal that the Fermi arcs extrapolate to nodal points at zero temperature.[27] However, the observation of Fermi arcs may be compatible with a reconstructed Fermi surface if the rest of the Fermi surface (the other side of the hole pockets, as well as the electron pockets) are obscured by inelastic scattering and matrix element effects. Another possibility is that the quantum oscillation measurements are probing a different, high magnetic field state, and not the zero field state probed by ARPES. For example, it has been proposed that antiferromagnetism might be induced by a field and lead to the low oscillation frequency observed.[31]

At face value, the SdH and dHvA measurements suggest that even the underdoped cuprates might be explained within a Fermi-liquid picture. However, this suggestion is controversial. First, the measurements can only be simply explained within a Fermi-liquid picture if there is symmetry breaking order or near-order. Second, this directly contradicts the suggestion that the pseudogap phase is a nodal liquid and that the Fermi arcs observed by ARPES extrapolate to nodal points at zero temperature. Very recently, Varma[32] has proposed that the quantum oscillation measurements might be compatible with a nodal liquid.

More work is needed to fully understand the implications of observing quantum oscillations in the pseudogap phase and to distinguish between the various possibile proposals for reconciling these data with the ARPES results. Studies on cuprate materials with different elastic and inelastic scattering rates, as well as different experimental probes, could shed light on reconciling the ARPES and the SdH and dHvA measurements. For example, Varma[32] suggests infrared absorption measurements to distinquish between a nodal liquid and a reconstructed Fermi surface. Also, the transition implied by Fermi surface reconstruction, whether it is induced by doping or by magnetic field, should show up in other experimental probes.

The nature of the pseudogap phase is still an open question despite more than a decade of intense effort focussed on this one phase. There is convincing evidence for both precursor order (or fluctuations) with a T* which hugs the superconducting dome, and for competing order (or fluctuations) with a T* which cuts through the superconducting dome. The latter is expected to end in a zero temperature quantum critical point under the superconducting dome. In fact, both of these phenomena

can sometimes be seen in a single experiment. For example, scanning tunneling measurements see both a "pairing temperature" and a pseudogap temperature above T_c.[33] This and the fact that multiple types of order or quasi-order are observed in at least some materials in some regions of the pseudogap phase, have complicated the identification of the key and universal features of the low-temperature pseudogap phase.

Nevertheless, the existence of a quantum critical point under the superconducting dome, even if precursor effects are also present, would seem to be a key ingredient to understanding high temperature superconductivity, and indeed a variety of proposals exist for the nature of the phases separated by such a quantum critical point. Most of these suggest a non-Fermi liquid state on the low-doping side, which makes the development of a complete theory of high temperature superconductivity particularly challenging. Our conventional theoretical formalisms of BCS theory and beyond break down for non-Fermi liquid states and, while our physical understanding of non-Fermi liquid states has deepened considerably and detailed models and calculations exist for highly correlated insulating states, our ability to calculate properties of metallic non-Fermi liquid states, except in one-dimension, is still very limited. A breakthrough in this area of theoretical physics, might finally allow a complete and predictive theory of high temperature superconductivity.

Given the intense effort and many ideas with strong supporting experimental evidence, it seems likely that the key to the pseudogap phase lies in one of the already existing theories. Certainly many individuals believe this is the case, but they do not all agree on which theory it is. As is already clear from just the one class of experiments discussed in detail above, further experiments are likely to confirm or rule out some of the possibilities. Consequently, this remains a very active area with the hope that new experiments, together with further advances in developing a robust theoretical framework which allows a thorough investigation of metallic non-Fermi liquid states, will resolve open questions in the not too distant future.

3.2. *Iron pnictide superconductors*

High temperature superconductivity was discovered in the iron pnictides just last year. In February 2008, superconductivity at 26K was discovered in $LaO_{1-x}F_xFeAs$,[34] which quickly rose to 43K in $SmO_{1-x}F_xFeAs$,[35] and 55K in $PrO_{1-x}F_xFeAs$.[36] Again, many different materials belong to the class of superconducting iron pnictides. They fall into two families, referred to as 111 and 122 because of their chemical composition; i.e. LiFeAS and ROFeAs, where R=Ce,Pr,Nd,Sm,... are 111's and $(A,K)Fe_2As_2$, A=Ba,Sr are examples of 122's. These materials, while containing no Cu, have many similarities to the cuprates and the key question right now is just how similar the pnictides and the cuprates are. In other words, is the physics of the high temperature superconductivity in the iron pnictides essentially the same as in the cuprates or, are the differences sufficiently important that a new route to high temperature superconductivity has been discovered? In either

case, assuming that electron correlations play a key role in the iron pnictides, as seems most likely, these are extremely interesting materials, and the effort expended per unit time on studying these materials has been even more intense than for the cuprates. In part, this is because the community has developed many highly relevant tools (both in theory and experiment) from investigations of the cuprates, which can now be quickly redeployed toward the iron pnictides. For example, in the early days of cuprate research, ARPES was unable to give definitive information, but the precision of ARPES has improved to the point where it is now a central tool for the investigation of high temperature superconductivity.

As mentioned above, the iron pnictides have much in common with the cuprates. They are both layered materials, with the FeAs layers playing the same role as the CuO_2 layers. Both involve d-electrons (from either Fe or Cu) playing a key role; both have antiferromagnetism and superconductivity in close proximity; and both are poor metals which become high temperature superconductors as the temperature is lowered or the doping is increased. However, there are also differences which may be important. One key difference is the band structure of the undoped compounds. The undoped cuprates have one electron per unit cell, so one is starting from a half filled band, which electron correlations turn into a Mott insulator. In contrast, the undoped iron pnictides have 6 electrons per unit cell which would be a band insulator if the bands did not overlap in energy. Because the bands do overlap in the pnictides, one is starting from multiple nearly filled or nearly empty bands. While there is evidence that the band structure is modified, perhaps even significantly, by electronic correlations, the undoped phase remains weakly conducting with five bands crossing the Fermi energy at zero doping. This band structure, calculated within local density functional theory,[37] agrees reasonably well with what is observed in ARPES experiments for LaOFeP,[38] as shown in Fig. 3. Recent work suggests that the effects of correlations may be more significant in LaOFeAs.[39]

From the band structure, one would expect electron correlations to play a much smaller role in the iron pnictides. In the cuprates, at one electron per site, the on-site Coloumb repulsion is extremely important and, in fact, leads to insulating behavior. In the iron pnictides, each band is almost empty or almost full, so within a band the electrons are far apart and the on-site intraband Coulomb repulsion is not so important. Interband electron interactions are also reduced because the wave functions are orthogonal. In a single band model, the Mott insulator transition as a function of onsite Coulomb repulsion, U, occurs roughly at the point where U is equal to the bandwidth. From the above arguments, in a multi-band model with nearly filled and empty bands, the critical U may be noticeably larger than the average bandwidth. However, there are other signatures, including the fact that the materials are poorer conductors than the band structure would suggest, which have led some to conclude that correlations do, in fact, play a very significant role and that one may be in close proximity to a Mott insulating phase, even though an insulating phase does not appear in the physical phase diagram of these materials. Furthermore, these materials display commensurate magnetism, which suggests

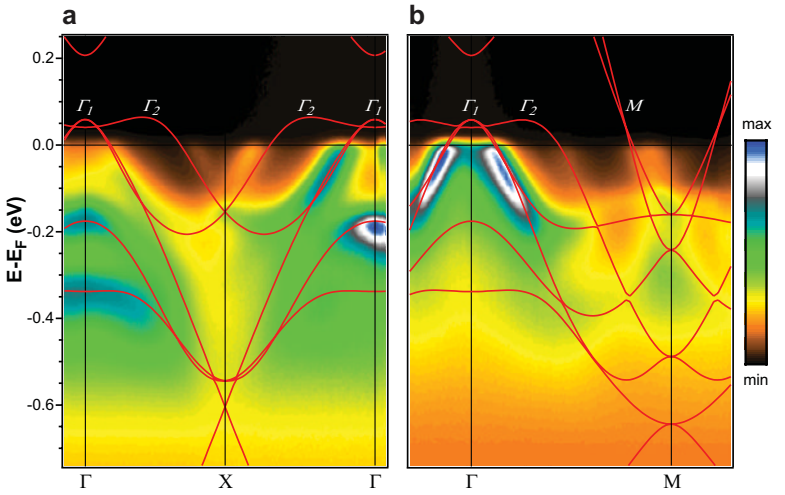


Fig. 3. ARPES data from LaOFeP compared with LDA band structure calculations which have been shifted up by 0.11eV and renormalized by a factor of 2.2 (red lines). From Ref. 38.

strong correlations.

As mentioned above, antiferromagnetism and superconductivity exist in close proximity, even co-existing in some of the iron pnictide materials. In contrast to the cuprates, the antiferromagnetism is itinerant, although it is also commensurate and appears at $(\pi,0)$.[40] The observed moment is small, typically less than $0.5\mu_B$, compared to the $2.5\mu_B$ expected from Hund's rule.[40,41] The role of Fermi surface nesting and localized exchange interactions in the observed magnetism is still an open question. The moment is reduced by more than one would expect simply from quantum fluctuations, and it has been argued that the small moment could arise due to combined effects of spin-orbit, monoclinic distortion and p-d hybridization.[42]

The symmetry of the superconducting gap can give important information about electronic correlations and the pairing mechanism. Given the close proximity of superconductivity and antiferromagnetism in these materials, it is natural to think that antiferromagnetic fluctuations might be driving the superconductivity. In the cuprates, the strong on-site Coloumb repulsion both stabilizes the antiferromagnetic insulating phase and drives the d-wave symmetry of the superconducting order parameter.[43] In general, higher angular momentum pairing is typically a signature of relevant repulsive interactions.

Clearly, there is great interest in determining the symmetry of the superconducting order parameter for the iron pnictides. To date, there is both evidence for nodes in the gap and for an isotropic (s-wave) gap. In particular, ARPES measurements have been taken as evidence for an isotropic gap on all five Fermi surfaces in one of the 122 compounds.[44] Thermodynamic measurements have seen power law behavior which is taken as evidence for nodes, although one needs to exercise cau-

tion in multiband materials. The superconducting gap can be quite small on some of the bands, making it difficult to distinguish between s-wave and higher angular momentum pairing. In fact, specific heat data was shown to fit a two-gap model well.[45] At this moment in time, the data appears to point toward s-wave in the Fe superconductors, but we do not yet have the definitive measurements that exist in the cuprates, in particular, the phase sensitive measurements. NMR gives evidence of singlet pairing, again compatible with s-wave pairing.[46]

Realistic calculations appear to have ruled out a pure phonon mechanism for the iron pnictides.[47] However, Tesanovic has proposed a combined phonon and electronic mechanism, where the phonons provide the attraction within a band and the bands are coupled through electron-electron interactions.[48] In this theory, the interband interactions set T_c, much like a Josephson coupling between phonon-mediated superconducting layers would set T_c. While this theory predicts s-wave gaps, the sign of the gap may differ on different bands. In principle, one can search for this "sign-effect" experimentally. Finally, there are many purely electronic theories, some which start from an intinerant state with spin fluctuations mediating the superconductivity and others which start with localized moments, a large on-site repulsion and proximity to a Mott insulator, so that much of what we have learned from the cuprates can be applied.

In summary, this field is still very new and is still changing rapidly. There is currently no consensus on the key question posed here: is high temperature superconductivity in these materials essentially the same as or different from superconductivity in the cuprates. Directly connected to this question is whether an on-site Coulomb repulsion plays a key role in stabilizing the magnetism and the superconductivity, as it does in the cuprates. This field is moving more rapidly than the cuprate studies in the early days because we have many more accurate techniques and probes, in theory and in experiment, available to us. However, one still needs high quality samples for many investigations and creating high quality materials is a mixture of science and art and takes time. Single crystals have recently become available, but one can expect further advances to be made in removing sources of inhomogeneity and disorder from the crystals.

3.3. *Strontium ruthenate*

Superconductivity in strontium ruthenate, Sr_2RuO_4 was discovered in 1994.[49] The transition temperature, T_c, is low, only 1.5K, but interest in this material stems from the fact that the superconductivity is believed to have a chiral p-wave order which spontaneously breaks time reversal symmetry. Such chiral order would be a solid state analogue of the A phase of He-3 and would also imply a topological order with the potential for exotic physics relevant to quantum computing, as discussed below. For this reason, much of the effort on Sr_2RuO_4 focusses on unambiguously determining the nature of the order parameter. While there exists strong evidence for chiral p-wave order, some inconsistencies and puzzles remain, as will

be discussed.

Sr_2RuO_4 is another quasi-two dimensional material with the same crystal structure as the cuprates. The electronic action takes place in the RuO_2 layers. Three bands cross the Fermi energy, and one of these (the γ band composed of d_{xy} orbitals) is believed to nucleate the superconductivity, with induced superconductivity on the other two bands.[50] The transition temperature, T_c, is sensitive to disorder, which immediately suggests that the pairing is likely to be of the unconventional (non-s-wave) type for which scattering around the Fermi surface can average the gap to zero. Furthermore, early NMR measurements of the Knight shift found that the spin susceptibility was unchanged as the temperature varied through T_c.[51] This is in contrast to the behavior expected for a conventional s-wave superconductor, where the spin susceptibility falls off rapidly below T_c as the spins condense into singlets. Therefore, the NMR results point to triplet pairing, of which the simplest possibility is a p-wave order parameter, although f-wave has not been ruled out. About the same time as the NMR results, muon spin resonance (muSR) experiments measured an additional muon spin relaxation which rises from zero at T_c and which achieves a maximum value as T approaches zero.[52] This extra relaxation was found to correspond to inhomogeneous internal fields with a characteristic strength of a few Gauss. Since these internal fields are zero above T_c, this experiment points to spontaneous time reversal symmetry breaking in the superconducting state. Recent experiments found the onset of the extra relaxation tracks T_c as T_c is varied by increasing disorder, reinforcing the interpretation that the time reversal symmetry breaking is directly associated with the superconducting state.[53]

With the experiments pointing toward a triplet, p-wave superconductor with broken time reversal symmetry, the question is which p-wave order parameters are compatible with the symmetry of strontium ruthenate. There are many allowed p-wave order parameters, as summarized in table IV in Mackenzie and Maeno.[49] In zero magnetic field, one can assume that non-unitary order parameters have higher energy, since they break the symmetry between up and down spins. Of the unitary p-wave order parameters, there is only one which breaks time reversal symmetry. It has an isotropic gap around the Fermi surface so it is energetically favorable because of the large condensation energy.

The order parameter for a triplet superconductor must specify the pairing amplitude for each of the three spin states and this can be expressed in terms of a d-vector which contains information about the symmetry of the gap and orientation of the spins:

$$\Delta(\mathbf{k}) = i(\mathbf{d}(\mathbf{k}) \cdot \vec{\sigma})\}\sigma_y \tag{1}$$

where the components of $\vec{\sigma}$ are the Pauli matrices. For unitary ($\mathbf{d} \times \mathbf{d}^* = 0$) states, the spin is zero along the direction of $\mathbf{d}$. The unitary p-wave state with broken time reversal symmetry corresponds to $\mathbf{d} = \Delta_0(k_x \pm ik_y)\hat{\mathbf{z}}$, which has a chirality given by the $\pm$ sign. The two chiralities are degenerate, so there is the possibility of domain structures. Due to spin-orbit coupling in strontium ruthenate, the d-vector

is oriented along the c-axis (chosen to be the z-axis here) so the spins are in the $S_z = 0$ state. This also corresponds to equal spin pairing ($\uparrow\uparrow$ and $\downarrow\downarrow$) in the ab (or xy) plane. In this state, each Cooper pair carries angular momentum plus or minus one, depending on the chirality, along the z-axis. The BCS wave function carries a total angular momentum of $N\hbar/2$, where N is the total number of electrons.

The BCS state described by this chiral p-wave order parameter is a two-dimensional analog of the A phase of He-3,[54] and is also closely related to the Moore-Read state proposed for a quantum Hall system at 5/2 filling.[55] As shown by Moore and Read, the 5/2 state has a topological order and supports Majorana zero modes at the edges and at vortex cores. Majorana fermions are their own antiparticle (i.e., $\gamma^\dagger = \gamma$, where $\gamma^\dagger$ creates a Majorana fermion) and two Majorana fermions are required to create an ordinary fermion, such as an electron. Much exotic physics, including non-Abelian statistics follows from the fact that this state supports Majorana fermions.

Even if strontium ruthenate does support a chiral p-wave state, the exotic physics is not immediately accessible because the direct correspondence is between the Moore-Read 5/2 state and a *spinless* (or, equivalently, spin polarized) chiral p-wave superconducting state. The equal spin pairing state appropriate for strontium ruthenate is equivalent to two copies (spin up and spin down) of the Moore-Read state and, consequently, supports two Majorana zero modes at the edges and at vortex cores and much of the exotic physics is lost. However, if the d-vector can be rotated into the ab-plane, and is free to rotate in that plane, the exotic physics predicted in the Moore-Read state becomes accessible.

A d-vector which is free to rotate in the ab-plane corresponds to pairing in only a single spin channel ($\uparrow\uparrow$ or $\downarrow\downarrow$) which suggests it might be stabilized by an external magnetic field. In fact, recent NMR experiments have been interpreted as evidence for such a state. Earlier NMR experiments were done with a magnetic field in the ab-plane and saw no suppression of the spin susceptibility below T_c, as one would expect for a triplet state with equal spin pairing in the ab-plane. However, more recent NMR experiments with the magnetic field along the c-axis also found no suppression of the spin susceptibility below T_c.[56] This is not compatible with a $S_c = 0$ state and has been taken as evidence that modest fields (less than 500G) are sufficient to rotate the d-vector into the plane. In He-3, which is isotropic, it is known that magnetic fields rotate the d-vector perpendicular to the field. The spin-orbit coupling in strontium ruthenate is sufficiently strong that it is surprising such low fields would reorient the spins. On the other hand, one needs to compare the energies of the different states in the presence of a field. It has been argued that there is, in fact, an energetically competitive state with the d-vector in the plane.[57] However, this state is non-chiral and, consequently, would not support the exotic physics of the Moore-Read state. Currently, it is an open question as to what state is stabilized in a c-axis field. Nevertheless, theorists have explored the possibility of exotic physics if a chiral p-wave state with a d-vector in the ab-plane is stabilized, so let me briefly review some of the highlights of these explorations.

If the d-vector lies in the ab-plane and is free to rotate, the system can support half-quantum vortices. The wave function acquires a phase of π if the structure of the vortex is such that the d-vector winds around the vortex core. Therefore, the orbital part of the wave function also only needs to acquire a phase of π, rather than the usual 2π associated with a vortex, for the entire wave function to be single valued. This corresponds to the Bohm-Aharonov phase of a Cooper pair circling half of the usual superconducting flux quantum, or $hc/4e$. Whether such a half-quantum vortex has a lower or higher energy than the regular vortex, depends on microscopics, and there have been proposals for stabilizing such vortices.[58] One can show that the half-quantum vortex supports a single Majorana zero mode bound at the core.[59,60] (This is in contrast to the usual vortex which has two zero modes in the core.) Furthermore, these half-quantum vortices obey non-Abelian statistics when one vortex is moved around another such vortex.[60] Non-Abelian statistics is exactly what is required in quantum computing, as the non-trivial winding connects distinct, but degenerate ground states with topological stability.[61] Of course, even if strontium ruthenate does support exotic vortices, one needs to carefully consider the role of the third dimension as one would expect the Majorana fermions to form a band along the c-axis, which will complicate their role in quantum computing.

Having presented evidence for chiral p-wave superconductivity and discussed some of the possible exotic physics which could arise from this state, I now want to turn to the more recent experiments which have provided both further compelling evidence for chiral p-wave order, as well as results which suggest otherwise. In particular, I will focus on the polar Kerr effect and the search for spontaneous edge currents.

In the polar Kerr effect, linearly polarized light is normally reflected from the sample surface as elliptically polarized light with a rotation of the polarization axis being the Kerr angle. One observes a non-zero Kerr angle if either left or right circularly polarized light is preferentially absorbed by the sample, as would be the case in a ferromagnet or a chiral p-wave superconductor. Kapitulnik's group observed a non-zero Kerr angle grow up as strontium ruthenate was cooled below T_c.[62] The Kerr angle rose from zero at T_c to a maximum of 60 nrads at the lowest temperatures. The sign of the Kerr angle, but not the magnitude, was affected by cooling in fields up to 100G. These data are qualitatively as expected for a chiral p-wave superconductor with a domain size larger than the beam size of incident light. In some runs a reduced Kerr angle was observed, which suggests the domains are not too much larger than the beam size which is about 25 to 50 microns across.

In a clean chiral p-wave superconductor the idealized Kerr angle is strictly zero from translational symmetry.[60] However, since the beam size is finite, one is not probing the system at strictly zero wave vector, and, in fact, a clean chiral p-wave superconductor displays interesting and nontrivial behavior at finite wave vector.[63,64] However, the beam is large enough in Kapitulnik's experiment that these effects should be negligible. Recently, Goryo showed that the lowest order impurity induced contribution to the Kerr angle comes from so-called skew-scattering

diagrams, which contribute in order $n_i U^3$, rather than the usual $n_i U^2$ term, where n_i is the density of impurities and U is related to the strength of the impurity potential.[65] Estimates of the Kerr angle from this impurity scattering model are smaller than, but comparable to, the observed value, if one takes somewhat optimistically large estimates for the density and strength of impurity scattering. Therefore, this seems like a possible, although somewhat marginal, explanation of the experiments. This theory could be tested by further experiments, since it predicts an unusual ω^{-4} frequency dependence for the Kerr angle. Furthermore, one could try increasing the amount of disorder, while still maintaining superconductivity (at a reduced T_c) to test this interpretation. Nevertheless, while some questions remain, the Kerr effect is a very direct probe of time reversal symmetry breaking and chirality and these experiments significantly strengthen the case for chiral p-wave superconductivity. As a final point, it is interesting that Goryo's theory only gives a non-zero result for p-wave and would give zero for a chiral f-wave superconductor.[65]

Another direct test for chiral p-wave order is to search for spontaneous supercurrents flowing at the sample edges and/or at domain walls.[66] In fact, the early muSR experiments are interpreted as evidence for supercurrents at domains walls in the bulk, since the magnetic field inside a single-domain chiral p-wave superconductor vanishes (except at defects which suppress the superconductivity, such as at impurity sites). The topological nature of the state, requires special edge modes at zero energy, but in addition, a chiral p-wave state supports a band of edge modes which carry a spontaneous supercurrent related to the total angular momentum of the state.[67] This supercurrent is localized roughly within a coherence length of the surface and is screened by an equal and opposite current within roughly the coherence length plus the penetration depth. Consequently, in the absence of domains, the field is strictly zero in the bulk, but there is a net magnetization or field localized at the surface. Similar currents flow at domain wall boundaries.[68] One should be able to detect the fields associated with these currents at the edges or from domain walls intersecting the surface, using scanning SQUID microscopy or a scanning Hall probe. Both techniques have been employed on strontium ruthenate, and no evidence of fields at the surface were observed.[69,70] Fig. 4, for example, shows the experimentally observed flux as one scans across the sample compared to the flux expected for a somewhat idealized chiral p-wave superconductor. The expected flux is about two orders of magnitude larger than the experimental noise limit. In fact, the experimental data can be well modeled by an s-wave superconductor screening a residual external field of 3 nT.

These null results are quite surprising and difficult to reconcile with chiral p-wave order. Very small domains (at the surface) could explain the null results because of the finite size of the pickup loops (8 microns for the SQUID and 0.5 microns for the Hall bar). Such small domains, roughly a micron or smaller, would be incompatible with the measured Kerr angle. Furthermore, domain walls cost energy and are expected to be present at low temperatures only due to pinning effects. Rough or pairbreaking surfaces, as well as other modifications to the theory, can reduce the

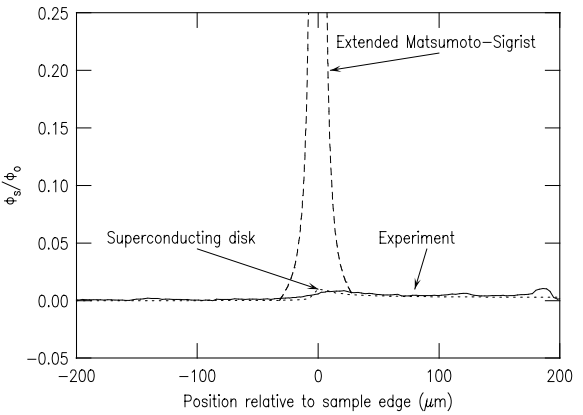


Fig. 4. SQUID scan across the edge of an ab face of a Sr_2RuO_4 crystal (solid line). The dotted line is the prediction for an s-wave superconducting disk in a uniform residual eld of 3 nT. The dashed line is the prediction for a single domain $p_x + ip_y$ superconductor, following the theory Matsumoto and Sigrist,[68] but modifed for a finite sample. The peak value of the dashed line is 1 (off-scale). From Kirtley *et al.*, Ref. 70.

expected signal, but no plausible explanation has been found which both reduces the signal to below the experimental sensitivity and leaves the interpretation of the positive experiments, such as the Kerr effect and muSR measurements, intact.[71,72] At the moment this is a puzzle, and it is interesting to note that the same puzzle persists for the A phase of He-3. The symmetry of the A phase has been established without a doubt, due to high precision measurements of the various collective modes, for example.[54] However, the mass supercurrents expected at the surface have never been observed, although in He-3, the d-vector is free to rotate and may do so near the boundary, which could suppress these currents.

The absence of observed currents in He-3 led Leggett to suggest an alternative to the BCS wave function.[73] While the BCS wave function carries an angular momentum of $N\hbar/2$, for Leggett's wave function this is reduced by a factor of $(\Delta/E_F)^2$. This would certainly make the supercurrents unobservable, although it would also eliminate the explanation of the muSR results in terms of fields associated with domain walls. However, I believe it would leave the Kerr effect interpretation intact. In any case, I think it still remains to be understood whether the weak-coupling limit of a chiral p-wave superconductor is described by the BCS wave function or an alternative, such as Leggett's wave function. For the case of s-wave, the two wave functions are identical.

In summary, there is compelling evidence pointing toward chiral p-wave order in the superconducting state of strontium ruthenate. In addition to the muSR and Kerr effect results discussed above, there are also several tunneling results which point toward chiral p-wave order.[74,75] The absence of observed edge currents remains a puzzle which is difficult to reconcile with chiral p-wave order. Furthermore, if one looks closely at the details of the various experiments, one finds that all the

experiments are relying on certain assumptions about domain sizes. Some experiments (such as the Kerr effect) require the domain size to be sufficiently large to interpret the experiment as evidence for chiral p-wave order, whereas other experiments require the domains to be sufficiently small (e.g. muSR). Consequently, the experiments are not as consistent with each other as one might first assume, and ideally one would like to be able to probe the domain walls directly, if they do exist. Certainly more work needs to be done to unambiguously determine the symmetry of the superconducting order. The striking observations of time reversal symmetry breaking, together with theories which point toward exotic physics and potential applications to quantum computing, provide significant motivation for further studies on this material.

4. Frustrated Magnets

The great challenge driving the search for new frustrated magnetic materials is to discover a material which supports a two or three dimensional quantum spin liquid. Spin liquids are ubiquitous in one-dimensional magnetic systems since quantum fluctuations prevent order. So far, spin liquids in higher dimensions have remain elusive in real materials despite an aggressive search over the last two decades. Theoretical models exist in higher dimensions, both for gapped spin liquids, which exhibit topological order, and gapless spin liquids which may have a spinon Fermi surface.[1] This field, including the relevant experiments, is reviewed by Subir Sachdev, so I will keep my discussion brief and just highlight a few points specific to the real materials currently under investigation. The best experimental candidates are typically spin 1/2 systems, for which quantum effects are maximized, either with geometric frustration and macroscopic classical degeneracy, or with proximity to a metal-insulator transition so that fluctuations, in particular ring exchanges, are important. Examples of the first kind include Herbertsmithite ($ZnCu_3(OH)_6Cl_2$), Volborthite ($Cu_3V_2O_7(OH)_2H_2O$), and Vesignieite ($BaCu_3V_2O_8(OH)_2$), which all correspond to spin 1/2 on a Kagome lattice. Examples of the second kind include the organics κ-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$ and EtMe$_3$Sb[Pd(dmit)$_2$]$_2$, with spin 1/2 on a triangular lattice, and close to a Mott insulator transition. All of these systems, as well as many more examples, exhibit no conventional magnetic order down to the lowest temperatures studied, often several orders of magnitude below the Curie-Weiss temperature which is inferred from high temperature susceptibility measurements.

While theoretical studies suggest either frustration and degeneracy or proximity to a Mott transition as conditions conducive to spin liquid behavior, real materials typically present challenges which complicate the search for a spin liquid. In particular, in materials which rely on geometric frustration and degeneracy, one seldom achieves the perfect or near perfect lattice structure. Typically the lattices are either distorted from the ideal lattice configuration or there is intrinsic disorder which is difficult (perhaps even impossible) to eliminate or both of these effects occur. If one runs through the long list of materials based on quantum spins on Kagome or

pyrochlore lattices, it seems that one can obtain clean materials with distorted geometry or disordered materials with ideal geometry, but, at least to date, not clean materials with ideal geometry. In other words, it seems that nature at least partially lifts the macroscopic degeneracy through either spontaneous distortion or disorder, rather than through the quantum fluctuations which would lead to a uniform spin liquid. For example, in Herbertsmithite, the Cu atoms form Kagome layers, but there is noticeable exchange of Zn atoms, which sit between the layers, and the Cu atoms. This is seen in NMR where one observes two different O sites, depending on whether one of the neighboring Cu is replaced by Zn or not.[76] Such disorder reduces the frustration, lifts the classical degeneracy and can affect many spins. At best, this can make the identification of the spin liquid state difficult and at worst, it can lead to a more conventional, but disordered, state. Nevertheless, materials discoveries often surprise us, and one may yet discover a material with a more ideal Kagome structure.

The second route to a spin liquid, proximity to a Mott insulating transition, does not rely on an underlying macroscopic classical degeneracy. The examples of organic compounds, mentioned above, have spins on a triangular lattice. Here, the spin 1/2's reside on large molecules, and disorder may also present problems but it does not play the role of partially lifting a necessary condition for the route to a spin liquid. For this reason, this second class of materials, which includes κ-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$, may be particularly promising materials for finding a spin liquid in two or three dimensions.

5. Topological Insulators

The discovery of the quantum Hall effect in 1979 ultimately opened up a new field of study which connects all the topics discussed here, namely, the field of topological order. The integer quantum Hall state is an example of a topological state which has no conventional broken symmetry and is not described by a local order parameter, but, rather, is characterized by a topological invariant, the first Chern number. Recent advances related to the quantum Hall effect are reviewed by Ady Stern.[2] Here I discuss recent experimental discoveries of the quantum spin Hall effect and topological insulators.

The quantum spin Hall (QSH) state was first predicted in 2005,[77,78] and discovered experimentally in 2007.[79] It is a topological state, closely related to the integer quantum Hall state but it does not require an external magnetic field and, in fact, arises in systems with time reversal symmetry. It occurs in two-dimensional systems with a non-trivial band structure arising from strong spin-orbit interactions, such that the system is an insulator in the bulk but supports topologically protected edge states. These edge states are analogous to the chiral edge states in the integer quantum Hall effect, and the QSH state can be thought of as two copies of quantum Hall states, one for each spin component, which move in opposite directions.

The QSH state, a new state of matter, was observed in HgTe quantum wells

surrounded by CdTe,[79] as predicted by theory. The observed conductance is independent of the width of the well, as expected for a conductance due to edge states only. Furthermore, its magnitude at low temperatures is the expected quantized value of $2e^2/h$, provided the sample is not too long (along the direction of the edge currents). In long samples, the conductance is suppressed, as the QSH currents are only protected by time-reversal symmetry. Molenkamp and coworkers[79] also verified that the QSH effect was destroyed by applying a magnetic field.

The QSH effect is not restricted to two dimensions and topological insulators also exist in three dimensions.[80] Again, these are systems with a non-trivial band structure due to strong spin orbit interactions and which support conducting, topological edge states. These insulators are distinguished from ordinary insulators by a Z_2 quantum number which takes the value $\nu = 0$ for ordinary insulators and $\nu = 1$ for topological insulators.

The key difference between ordinary and topological insulators can be understood by focussing on the properties of the edge states. While an ordinary band insulator can support edge (or surface) states, these edge states are not topologically protected, any crossings (degeneracies at the same point in k-space) typically occur at general points in the Brillouin zone, and perturbations will open up a gap at these crossings. In topological insulators, the special edge states occur at symmetry points (actually at Kramers degeneracy points, such as Γ and M). Time reversal symmetry requires that these surface states come in Kramers pairs and protects them against perturbations. One can have a single Kramers pair at these symmetry points. It follows that topological insulators ($\nu = 1$) have an odd number of surface states crossing the Fermi energy between the points Γ and M, say, whereas this number must be even for a conventional band insulator. This has been observed in $Bi_{1-x}Sb_x$ in a beautiful set of experiments.[81] High resolution ARPES measurements found 5 surface states crossing the Fermi between the Γ and M points. These data are shown in Fig. 5. Care has been taken to identify the surface bands, accounting for multiple bands which are close by in energy, by observing the splittings at other points in k-space, and these measurements provide compelling evidence of a three-dimensional topological insulator. More recently, spin-ARPES was used to probe the spin degrees of freedom and confirm the chirality of the surface states.[82]

In striking contrast to the field of novel superconductors discussed above, one point that stands out in the field of topological insulators is the detailed, predictive power of theory. Theorists predicted specific materials to be candidates for topological order, which were then experimentally verified a short time later. This, of course, is because the novel physics of topological insulators occurs at the non-interacting or one-electron level. In fact, in the above discussions and in most of the theoretical work, electron-electron interactions are ignored and assumed to be weak. Much less is known about potential strongly correlated topological insulators and whether there are spin analogues to the fractional quantum Hall effect, for example. This is currently an active field of study, as is the search for new physics in the non or weakly interacting topological insulators discussed here. There already are addi-

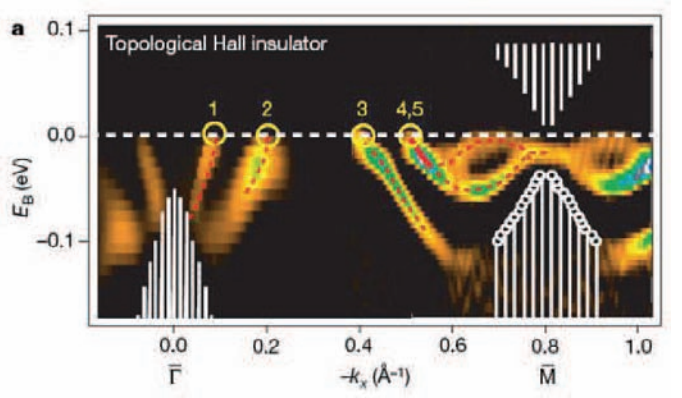


Fig. 5. ARPES data showing the surface band dispersion of $Bi_{0.9}Sb_{0.1}$ along Γ–M. The Fermi crossings of the surface state are denoted by yellow circles, with the band near $-k_x \approx 0.5\mathring{A}^{-1}$ counted twice owing to double degeneracy. From Ref. 81.

tional theoretical proposals, such as the search for an emergent magnetic monopole induced by external electric field,[83] which are awaiting experimental discovery.

6. Conclusions

Real materials support new topological states connected to (but distinct from) the quantum Hall effect, such as two and three dimensional quantum spin Hall and topological insulating systems. In addition, strontium ruthenate may support a chiral p-wave state which is also connected to a topological quantum Hall state. This illustrates the enormous influence the quantum Hall effect has had on condensed matter physics and the field of quantum and topological order. It will certainly be interesting to explore the possibility of fractionalization in these topological states, to see if the analogies with quantum Hall physics go even deeper.[84]

On the other hand, quantum ordered or topological states motivated by studies of the high temperature superconducting cuprates (rather than by quantum Hall studies) remain elusive in real materials in dimensions higher than one, despite intense efforts in discovering, creating and improving frustrated magnetic materials. There has been enormous progress in understanding the theory of quantum spin liquids, both gapped, topological spin liquids as well as gapless, quantum ordered spin liquids. Many frustrated magnetic materials exhibit correlated spin states at low temperatures with no magnetic order, often down to temperatures which are less than 10^{-4} of the Curie-Weiss temperature. However, while candidates for spin liquids exist, noteably herbertsmithite and the organic κ-$(BEDT\text{-}TTF)_2Cu_2(CN)_3$, a smoking gun experiment for spin liquid order remains elusive and often intrinsic disorder, lattice distortion or anisotropic interactions play a key role in differentiating the real materials from the theoretical models.

Superconductivity remains a fascinating and active area of research. Supercon-

ductivity shares a property with the other subjects discussed here, quantum spin liquids and topological insulators, in that even ordinary BCS superconductivity is a type of topological order.[85] However, more recently, we have seen that it may be able to support further topological order, such as chiral p-wave order in strontium ruthenate. The cuprates gives us an example of a superconductor with strong repulsive interactions playing a key role. In addition to providing us with the highest superconducting transition temperatures known to date, the cuprates also exhibit the intriguing but puzzling pseudogap phase and appear to support a robust quantum critical point which may be connected to much of the anomalous observed behavior. The discovery of a new class of high temperature superconductors has generated renewed interest, but the question remains whether these new iron-based superconductors will provide new insights into the phenomenon of high temperature superconductivity or whether they will instead generate a new set of puzzles of their own.

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Discussion

M. Rice Now, in organising the discussion today I thought it would be good if we started with a detailed discussion directly connected to Catherine's talk and then after the break we would have a more detailed and longer discussion on issues to do with high T_c superconductors, iron superconductors and strontium ruthenates. So first I would like to ask you for questions or comments directly related to the talk.

M. Cohen Nice talk, just a few comments on the iron superconductors and the theoretical data. On the electron-phonon interaction, we have not published anything yet, but we have done extensive calculations to really check out phonon induced pairing, and we just cannot get strong enough coupling. There are still people around who are claiming that the phonon coupling is going to be strong when other interactions are added in. The thing that is really peculiar, which you sort of mentioned, is that without fluorine in the iron-arsenide systems or without oxygen vacancies there is no superconductivity. You put in a pinch of fluorine and it becomes superconducting. The question is what does the fluorine do? The strange thing is that you would not expect the fluorine to affect the phonons much, particularly the low frequency phonons for these heavy atoms. However it turns out that it does, and all the virtual crystal models that have been calculated so far completely miss this. But if you make a super cell,and put the fluorine in, you find that the phonons are affected dramatically. You also find that the charge distribution is strange. You can also get doping different ways. It also turns out that there are some holes in the iron-arsenide layer. Early on, people were just getting electron doping. So the configuration of doping as you go from layer to layer to layer is really unusual. If you integrate over a plane and then calculate how many excess electrons and extra holes there are, you get some strange effects. I think these features are all interesting, and they may bear on what ultimately will be the correct theory.

C. Kallin I was worried from your comments on phonons, that sort of lend support to Tesanovic's idea that the fact that the fluorine is really affecting the phonons and that it induces superconductivity means that maybe phonons are playing a role but they are not sufficient to really give you the T_c's you observe. Are you supportive of that scenario?

M. Rice Marvin, I thought there were some materials that did not have fluorine. You have all these different layers in between the iron arsenide layers and not all of them have fluorine in but the superconductivity seems like in the cuprates not to be sensitive that much to what you put in between, just as long you dope it.

M. Cohen As I mentioned with oxygen, vacancies and other a pproaches or by looking at different materials, you can also obtain similar effects.

M. Rice But are there not these other materials with bismuth etc that do not have...

C. Kallin Yes, but I think his point is that it is evidence that phonons seem to matter at least under certain some circumstances, right? Even though fluorine may not be important but it is actually telling you something that the fluorine can help superconductivity and that it does change the phonons, so you might conclude that there is a connection between those...

M. Cohen Z.-X. Shen maybe will say something, because we find that phonons around 23 meV are highly affected even though the fluorines are so light. But I understand experimentally people are seeing things of this kind. Maybe we can hear from some of the experimentalists later.

M. Rice Do you want to take up on that? Chandra...

C. Varma I searched a couple of months ago whether there is any systematic data on specific heats and magnetic susceptibilities in these materials, if there was anything unusual or interesting. To my dismay I discovered that there was not even systematic data about these things. Then when I looked at the specific heat at the transition, at least what I found 2 months ago was that there was no indication in almost all of these compounds that there was bulk superconductivity. Now, I was talking to Patrick Lee yesterday and he told me that there is another of these series of a different crystal structure in which there is systematically a specific heat signature of the superconducting transition. The thing that you showed that has a 60-degree T_c, I believe, has a signature I checked but I have not seen anything before. So what I am trying to say it is that it is very hard to systematically think about these things unless we know what the normal state parameters are, and even know whether their bulk phases are superconducting.

C. Kallin Right, so you are saying that some of these compounds where they are claiming that they are superconducting are not necessarily bulk superconductors, is that your point?

M. Rice That is a pretty strong claim!

C. Varma Of the ones I had looked at, I had to conclude that they were not bulk superconducting.

C. Kallin Well, that makes me think of the comments I heard from experimentalists, I guess Ali Yazdani had some of the single crystals and he wanted to study them with his STM technique. What he found is that he could study the entire range of doping on that single crystal. Depending where he looked on the

surface he saw everything going from the low doping to the high doping and yet it was supposed to be a single crystal. It is early days for the samples but obviously there is information coming out about the band structure and gaps. However we have to be cautious about some of these experiments.

M. Rice Things like NMR are bulk probes and if they see real effects associated with the superconducting transition there, then it seems to me pretty clear it must be a bulk effect.

C. Kallin It could still be inhomogeneous I guess...

M. Cohen In a few days some of us will be in Beijing, and we will know more about the experimental situation. The claim is that there is better heat capacity data. However, as a matter of history, Berndt Matthias always said when new superconductors were discovered that there is not a proven bulk effect until you get good heat capacity data, but I'm convinced that there is enough evidence now to conclude that it is bulk superconductivity in the iron systems.

E. Shimshoni I have a comment on a different topic, about the topological insulator. This is also connected to something that Marvin said during yesterday's morning session. I believe that exactly the same physics or very similar physics is realized in graphene. There are some very nice experiments on zero doped graphene in strong magnetic fields which my collaborators and I believe can explain using exactly the same physics.

C. Kallin but in the case of graphene the spin orbit coupling is too small?

E. Shimshoni No, in that case it is not spin orbit. What happens is that in very strong magnetic fields the zero Landau level splits. You can have edge states which have spin up and spin down associated to chirality and that can give you transport properties that can be very diverse.

C. Kallin I see, that sounds very interesting, that may fit in with tomorrow's session too...

N. P. Ong I would like to return to the iron pnictides to discuss whether it is surface superconductivity or bulk. One item of interest is that we have looked at flux flow and the upper critical field in single crystals of the 1-2-2 pnictide grown in Beijing. The encouraging news is that the melting field is incredibly high, so you do not see any of the vortex liquid effects that we encounter in BSCCO, for example. The steepness of the melting field is even higher than in YBCO, so you easily get a hundred Tesla for the melting field projected to 4 Kelvin. This seems to me that the system is much less 2 dimensional and perhaps much more optimistic in terms of applications. The vortex solid is really there and hangs on even to very intense fields. The second comment is regarding the pairing symmetry. One way we can check this is to look at the quasi-particle population by measuring the thermal Hall conductivity. Again, in these 1-2-2 crystals, you see a huge thermal Hall conductivity indicative of nodal quasi-particles. So the transport and the thermodynamics seem to favour nodal quasi-particles whereas ARPES so far seems to favour an isotropic gap.

S. Davis When we look at those families, we see that there is an intense nanoscale electronic disorder and that it is linked to where the doped atoms are: specifically cobalt substituted for iron. There are lots of quasi-particles near the chemical potential but it is not clear that they are nodal quasi particles because there is so much disorder scattering in the environment that the states near the chemical potential are filled up. So I do not have a feeling as to the symmetry but I would caution any deduction about the symmetry based on a knowledge of the density of states near the chemical potential on the assumption that it is a homogeneous superconductor, because it is not.

N. P. Ong We should discuss that but the finding is that the mean free path really increases below T_c just like in cuprates so it is actually at first glance not compatible with disorder scattering.

Z. X. Shen On the surface issue of iron pnictides for any of the surface sensitive experiments, I think this issue has not been fully resolved. We work now on the first phase, in that case this is a 1-1-1-1 phase. There are two phases, one is the single layer phase. On the single layer phase clearly there is a surface charging, among the 5 pieces of Fermi surfaces. Four of them look pretty similar to what you would have expected and one of them is relatively flat. That is why any surface charge would make the topology change a lot, as [name unclear] did work out pretty well. On the arsenic compounds, the basic band structure looks more complex and did not agree with the LDA as well as the first phase. That is very interesting. It is also very unusual that the band structure shows a clear spreading at the phase transition, which we do not yet know whether it is due do the magnetic phase transition or a structural phase transition. Both happen. So before we conclude whether it is an s-wave or a d-wave, I think the surface issue needs to be resolved. I would say that for the 1-2-2 phase, where all the superconducting gaps were measured, we looked at the spin density wave states and it did not show that kind of spin density wave gap you would see. On the other hand, if the band structure spreading is about 60 meV, what one would expect is caused by the magnetism then I would say there is some local magnetic effects in play. But again let me say that since both structural and magnetic transitions happen at the same time, it is not resolved yet. So I would say before we conclude based on ARPES, what the symmetry of the order parameter is, the zeroth order is that you get the electron count right at the surface. Also the SDW gap was not that obvious, so I would say if you could not get the simpler case resolved, because that is a bigger effect, the transition temperature is higher, completely resolved, then the harder thing, the superconducting gap at lower temperature, needs to be taken with caution.

C. Kallin Good point.

L. Balents I guess I have a question and not a comment: as a theorist for me it seems somehow most important to understand to what extent Mott physics is playing a role in these materials in terms of similarities and differences to the cuprates. The small magnetic moment seen in the parent compounds would

seem to be an indication against that; naively one would think that there is a spin density wave perhaps, although one could hope to wriggle out of that. I guess the question is to what extent has the spin density wave scenario really been explored in comparison to experiments like what Z.X. Shen has just mentioned: measurement of the gap, was the gap consistent with the spin density wave gap, consistent with the Neel temperature, spin wave velocities,... is there any experimental information on that?

C. Kallin Yes, so maybe someone else has an answer...?

Z. X. Shen The specific answer to the question is that we looked very hard for the spin density wave gap and what we see is something more closer to a sort of local moment splitting rather than a band folding. If for the simple spin density wave the gap should have happened at a k_F, a folding k_F and we could not find that very obviously. On the other hand, if you look at any of these systems the band splitting at the magnetic and structural transitions is very very clear. So I think it is a pretty interesting unresolved issue.

L. Balents Part of the question comes from talking to Igor Mazin who has been doing first principles calculations on these things and claims that it is rather unusual that those calculations give a larger moment that has been seen in experiments and with that large moment one finds things consistent with the structure but without it one does not ...

C. Kallin Sorry, these are LDA calculations or?

L. Balents Yes.

C. Kallin But in others, like I guess Phillips has a paper where he claims with the spin orbit coupling together with hybridization with the arsenic, that he can get a small moment ...

L. Balents Well there are many ways for us to try to wriggle out of that.

M. Rice So, Patrick Lee as been the one who published on that, so...

Prepared Comment by Patrick Lee: Comments on Recent Advances in Iron Pnictides Research

1. Abstract

The recently discovered Fe Pnictide superconductors offer a new path to unconventional superconductivity other than the cuprates. At this early stage of research, a number of key issues remain open. I shall make some remarks on the strength of correlation effects, the size of mass enhancement and the existence of gap nodes.

2. Introduction

The discovery of "higher T_c" in Fe pnictides created great excitement in our community. One normally associates iron with ferromagnetism, making it the last place

one might look for superconductivity. Yet Nature once again presented us with a surprise gift which points to additional pathways to high T_c superconductivity other than the cuprates. Early data invite obvious comparison with the cuprates: superconductivity in both cases arises from doping of a parent compound with antiferromagnetic order. However, the similarity stops there. Unlike the cuprate whose parent compound is a Mott insulator, the FeAs system has an even number of carriers per unit cell and starts out as a metal with small electron and hole pockets. The antiferromagnetic order is a spin density wave, probably due to nesting of these pockets. Unlike the cuprate where a single Cu $d_{x^2-y^2}$ orbital is active, the iron system involves multiple orbitals at the Fermi level. The key question is how strong is the correlation. Is the parent compound sufficiently close to a correlation driven insulator that one should start with a local moment description of the iron ion, or does an itinerant picture suffice? What is the role of Hund's rule coupling which is absent in the cuprates? Finally, what is the pairing symmetry?

3. Strong Correlations or Not?

My first remark concerns the number of orbitals needed to describe the states near the Fermi energy. The LDA band structure shows two hole pockets near the Γ point and two electron pockets at the M point based on the traditional Brillouin zone (BZ) with two Fe per unit cell. We showed that the crystal has an additional symmetry, i.e., translation by the Fe-Fe distance followed by a reflection about the z axis, which allows a unique description of the band structure in the "expanded" BZ which corresponds to a single Fe per unit cell.[1] In this description the 2 hole pockets are still located at Γ, but the electron pockets are separately located near $(0,\pi)$ and $(\pi,0)$. The hole pockets are made up of d_{xz} and d_{yz} orbitals, while the electron pocket near $(0,\pi)$ and $(\pi,0)$ are made up of $d_{yz} - d_{xy}$ and $d_{xz} - d_{xy}$ orbitals, respectively. Thus 3 orbitals are the minimum needed to describe the Fermi surface wavefunctions. In the literature there have been attempts to describe the band with 2 orbitals, in which case the Fermi surface topology comes out wrong in that the hole pockets are separately located at Γ and (π,π). The wavefunction of the electron pocket is quite intricate in that it changes character from pure yx to pure xz as one goes around the pocket at $(0,\pi)$. We showed that such wave function change can give rise to sign changes in the effective interaction, which leads to nodes in the pairing order parameter.[1] In general, it is not easy to create nodes around such a small Fermi pocket, because one requires an effective interaction which is rapidly varying on the scale of the small Fermi momenta. The proper treatment of wavefunctions around the pocket is one way of generating such rapidly varying potential.

The question of the strength of correlation is also not settled unambiguously. Early work suggests that the antiferromagnetic order can be understood as spin density wave, but recent data show surprisingly large ordered moments of 0.8 μ_B which suggests that a more local picture of the moment may be appropriate. If

the moments are localized, one expects substantial mass enhancement because the carriers grow out of adding electrons and holes to the local moments and will be difficult to move.

To address the question of correlation one would like to have information on basic questions such as the size of γ (the coefficient of the linear T term in specific heat) which one can compare with band calculations to get an idea of the mass enhancement. Unfortunately, this is not an easy measurement to make, because superconductivity intervenes. Early data on polycrystalline samples did not give us a clue because the transition is very much smeared. A few months ago a modification of the original structure LaFeAsO was discovered, where the LaO layer is replaced by a Ba layer which can be doped by K, resulting in the chemical formula $Ba_{1-x}K_xFe_2As_2$ (called the 122 material). Unlike the original $LaFeAsO_{1-x}F_x$ (1111 material), large single crystals of the 122 material can be made. Note that while the two materials share the same active FeAs layer, this 122 compound is hole doped while the original 1111 is electron doped. (In another version of 122, doping is accomplished by substituting Co for Fe in the FeAs plane, resulting in $BaFe_{2-x}Co_xAs_2$. This version is electron doped but has substantial in-plane disorder.) Recently a large number of experiments were performed on the 122 $Ba_{1-x}K_xFe_2As_2$ crystals. These include a specific heat experiment[2] and three ARPES measurements.[3–5] The specific heat measurement by Mu *et al.* is particularly impressive in that for the first time a BCS-like jump in the specific heat is clearly observed at T_c (36 K). The jump is surprisingly large, $\approx$ 49 mJ/Fe-mol K^2. (I convert from the unit used in the paper mJ/mol K^2 to per Fe-mol for ease of comparison later.) The authors also measure a downward shift of T_c in a magnetic field up to 9 T, from which they extrapolate to obtain $H_{c2}(T = 0) = 100$ T. Using this value of H_{c2} they estimated $\gamma = 31.6$ mJ/Fe-mol K^2, which is consistent with the specific heat jump according to the usual BCS ratio.

The γ value is very large compared with 6.5 mJ/Fe-mol K^2 obtained from LDA calculations for the undoped LaFeAsO.[6] Since the FeAs layer is basically unchanged, this is a reasonable starting point for comparison with the 122 material. Does this imply a mass enhancement of 5 and therefore strong correlation? I think this conclusion is unwarranted in light of several ARPES experiments done on the same 122 material. The consensus seems to be that overall the bands pretty much follow the LDA dispersion but with a factor of 2 band narrowing. Wray *et al.*[5] directly measured the Fermi velocity of the inner hole pocket to be 0.7 eV Å. Using the measured areas of the inner and outer hole pockets and assuming they have the same Fermi velocity, I estimate that the two hole pockets account for a γ of 6 mJ/Fe-mol K^2. Since the LDA calculation included the contributions from two electron pockets as well, these numbers are consistent with roughly a factor 2 renormalization of the Fermi velocity, not a factor 5. The question is then where do the remaining 25 mJ/Fe-mol K^2 come from. The paper by Zabolotnyy *et al.*[4] found that the pockets near the M points are totally different from those predicted by LDA band calculations. Instead of roughly circular electron pockets they found elongated hole pockets

which they call propeller blades. The band bottom of these blades is only about 20 meV below the Fermi level. Thus it is possible that a large part of the disagreement with the LDA γ value comes from band structure effects, and it will be interesting to see if the complicated low energy band near the M point has a large enough density of states to account for the very large observed γ. This will be consistent with the suggestion of Mu *et al.* that the γ for hole doped 122 materials may be 3 to 5 times larger than the electron doped 1111 compounds. The conclusion that mass enhancement is at most a factor of two is also reached by direct measurement of the mass from quantum oscillators in LaFePO.[7]

One interesting consequence of the low Fermi velocity noted by Wray *et al.* is that the standard formula $\xi_0 = v_F/\pi\Delta_0$ implies a surprisingly small ξ_0 of 20 Å or less, making it comparable to that of the cuprates, despite a much smaller energy gap. (This is because v_F for the cuprate is larger, ≈ 1.65 eV Å.) The short coherence length gives $H_{c2}(0) = \phi_0/2\pi\xi_0^2 \approx 100$ T, consistent with that inferred from the specific heat data. Mu *et al.* also found that the H_{c2} anisotropy is modest. Together with the high $H_{c2}(0)$, this is good news for potential applications.

4. Gap Nodes and Pairing Symmetry

I end with a few comments about the issue of gap nodes. The ARPES paper by Ding *et al.*[3] reported roughly isotropic gaps. The specific heat data at low temperatures is fitted by a gap of 6 meV, consistent with the smaller of the two gaps seen by ARPES, suggesting that these are bulk properties. Furthermore, the specific heat is linear in H, in contrast with the $\sqrt{H}$ behavior measured by the same group on electron doped 1111 polycrystals earlier.[8] The latter was taken as evidence for nodes. Another strong evidence for nodes came from NMR measurement of $\frac{1}{T_1}$ in 1111 material, which fits T^3 law over almost 3 decades.[9] Thus, while the evidence is strong for the absence of gap nodes in the hole doped 122 material, the possibility that electron doped materials may be different remains open.

Finally, there is evidence that the Fe pnictides as a class may exhibit even more diverse behavior. A recent NMR paper on the original doped LaFePO material ($T_c \sim$ 8 K) by Nakai *et al.*[10] showed that its fundamental properties may be totally different. In the FeAs system, the Knight shift decreases by about a factor of two from room temperature to T_c. That in itself is a mystery. In FeP the Knight shift increases with decreasing temperature, suggestive of ferromagnetic fluctuations. Furthermore, its $\frac{1}{T_1}$ *increases* below T_c. I have not encountered this behavior in superconductors before and it seems hard to reconcile with spin singlet pairing. This raises some hope that this may be a good material to look for the triplet p pairing we predicted.[1] On the other hand, singlet pairing is quite well established on the electron doped 122 single crystal $BaFe_{2-x}Co_xAs_2$ by a Knight shift measurement.[11]

The availability of large single crystals has launched a new phase in the iron pnictide research. However, as is often the case in this line of work, it appears that things will get more complicated before they become simple.

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Discussion

M. Rice I would like to restrict the discussion now to these iron-arsenic compounds.

A. Stern Is there Nernst effect data on this iron compounds?

N.P. Ong Yes, there is one report from China, it is small. We ourselves have not started the Nernst experiments yet.

A. Stern But the data available does not show this large enhanced ...

N. P. Ong Frankly, I do not remember what the data said...

A. Georges Just a word supporting Patrick's claim that these may be intermediate coupling. There has been a couple of recent works aiming at calculating U from ab initio screened calculations, one in my group and one in Japan and we actually find that U is of the order of the iron bandwidth, so you know anywhere between 3 and 4 eV. All words of caution for this sort of calculations...

P. Lee There is a paper by Anisimov. First of all, he pointed out that the proper U that you should use, for a tight-binding model, you should use orbitals which are not the original atomic orbitals but Wannier orbitals which are much more extended because they are a mixture of the iron orbitals with the arsenic orbitals and when he does that he claims he gets an effective U for the effective tight binding orbital that is quite small, 0.8 eV. And an interesting thing is that the Hund's rule coupling is not renormalized down by as much so now he gets a situation where the U is actually not much bigger than the Hund's rule coupling.

That is why again I think the Hund's rule coupling may be playing a role.

A. Georges Well, I am actually not sure I agree with that. If you actually consider these very extended Wannier functions, which in these materials are particular extended, there is a lot of hybridization between the arsenic and the iron, you would actually find that if you look at the nearest neighbor Coulomb interaction,it is going to be pretty big as well. So, I think these down foldings onto these extended Wannier orbitals are a bit dangerous.

P. Lee So you get different numbers from him, you get bigger numbers.

A. Georges Oh well, the numbers depend on what is your Hilbert space of course.

P. Lee But if you fold down to these effective hopping on the iron...

A. Georges No, we get larger values...

P. Lee You still get larger values, ok!

M. Rice Any more comments or questions on iron-arsenic? Yes, Subir?

S. Sachdev I have a question on the magnetism: is there accurate enough data from neutrons to see that the magnetism is not exactly commensurate, as might be suggested by the spin density scenario, when you dope the system?

P. Lee I am sorry, I could not quite understand.

S. Sachdev The magnetism in the material from neutron scattering is at $(\pi\ ,0)$. Has that been tested very carefully as a function of doping whether it remains commensurate or the wave vector moves away to $2k_F$ as you might expect in a spin wave density picture?

P. Lee Oh, I think in this case if it is nesting the wavevector may not move because you are mixing a particle-hole and the centre of the particle-hole separation is sitting at $(\pi\ ,0)$, there is no obvious reason why it should move. But as far as data is concerned I am not aware of any careful study.

S. Sachdev But once you dope, even when you start doping? I agree the parent compound should be commensurate.

M. Rice It, may or may not move. For example, in chromium and its alloys, there is a whole range of electron/atom ratios which does not move the commensurate wavevector.

P. Lee The answer is that I do not know of any data.

M. Rice I would like now to continue the discussion and move on to the high T_c cuprates. One of the interesting developments in the last couple of years has been a series of really new experiments with some unexpected and surprising results. So to start the discussion, the first short presentation will be given by Chandra Varma about the time reversal breaking symmetry and orbital currents.

Prepared Comment by Chandra Varma: Scholia on a Remark by C. Kallin

Dr. Kallin in her talk mentioned that a time-reversal symmetry breaking phase which I had predicted has been observed to set in below the so called Pseudogap temperature $T^*(x)$, where x is the doping density, in several families of underdoped cuprates. I wish to expand on her remark and point out its significance.

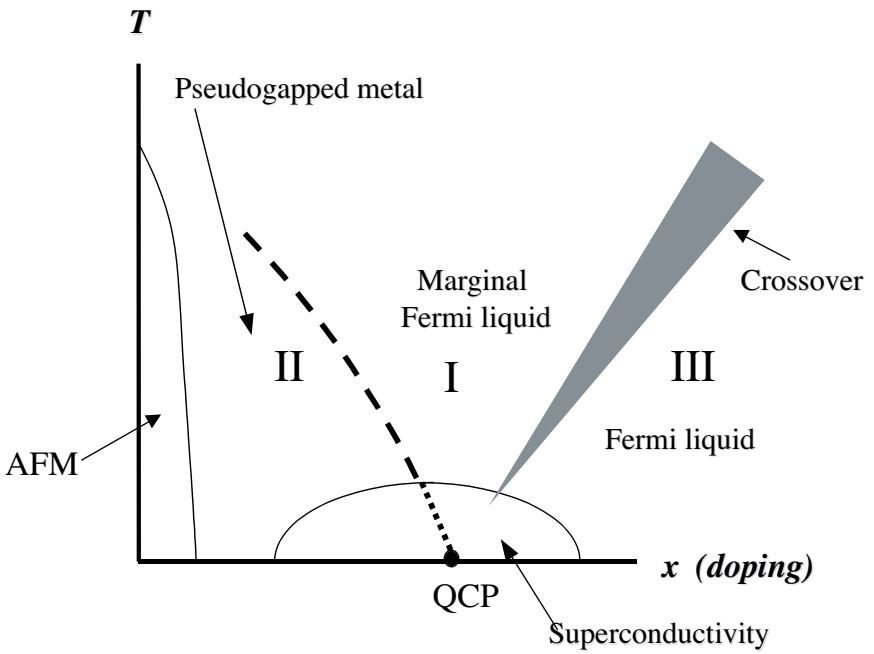

Fig. 1. Schematic Universal Phase diagram of the Cuprates with hole-doping. The lines drawn demarcate regions which mark changes in thermodynamic and transport properties in all Cuprates. The line marked T_p has been called T^* elsewhere.

Phase Diagram of the Cuprates: The generally (but not universally) accepted phase diagram of the Cuprates is shown in fig. (1). In Region I, anomalous but simply characterized power laws are seen in all transport properties. They were understood phenomenologically[1] as due to a scale invariant spectrum characteristic of a quantum critical point (QCP) residing within the superconducting dome. The quantum critical fluctuation spectra is unusual; it is spatially local and with a $1/t$ dependence temporally. This led to several predictions, most notably about the marginal fermi-liquid self-energy in single-particle spectra, which has been thoroughly verified and Raman spectra where the long-wavelength form of the quantum critical spectra is directly observed. The correspondence with experiments of the consequences of this hypothesis were so strong that it was natural, in keeping with the desire to find a single unifying principle for all universal properties of the

Cuprates, to tie the entire Cuprate phenomena to the physics of the QCP.

Further progress on the problem could come only from a microscopic theory which was motivated by the following considerations. Two lines usually emanate from a QCP, dividing the phase diagram into three parts: A part with a broken symmetry, a part where the correlations are dominated by quantum-critical fluctuations which is truly a region in which quantum correlations for $\omega/T \gtrsim 1$ give way to classical correlations for $\omega/T \lesssim 1$, and a third which is dominated by quantum correlations which for metals usually are correlations of a Fermi-liquid. This bears correspondence with Fig. (1). Indeed Fermi-liquid properties are found in Region III of the phase diagram. Moreover crossing from Region I to Region II across the line marked T_p is attended by a change in the temperature dependence of all thermodynamic and transport properties. However, there was no sharp change in any property observed and certainly no singularity in the specific heat at $T_p(x)$. And the myriad of experiments in the Curpates had revealed no broken symmetry below T_p. The changes observed going from Region I to Region II are however dramatic and led to asking if some unusual order parameter might arise ending up at a QCP as in Fig. (1).

Model for the Cuprates: Unravelling the nature of the QCP and the hidden order depended on having an adequate microscopic model for the Cuprates which could be examined for its possible phases. This is one of the knottiest and most contentious issues in the game with almost every theorist following Anderson's lead that the Hubbard model with its one *effective* degree of freedom per unit-cell must have all the physics. Opposed to this was the suggestion that the unique properties of the Cuprates arose because of the proximity of the Cu ionization and the oxygen affinity levels.[2] Under this condition, a Hubbard model is not an adequate model for the cuprates; a model with 3 orbital degrees of freedom in the unit-cell and nearest neighbor repulsions beside the local repulsions is necessary.

A plethora of phases is possible in such a model. The search could be restricted because any spin-rotational symmetry breaking phase or translational symmetry breaking phase of any kind appeared ruled out by experiments. This left only time-reversal breaking phases with two different arrangements of flux patterns in each unit-cell with different reflections and inversion symmetries and spontaneous broken spin-orbit rotation phases.[3] The latter could also be ruled out from existing experiments. This left the time-reversal odd phases with translational symmetry preserved as the only phases to be examined and suggested for experiments to discover. The mathematics of such phases, being in the space of three degrees of freedom per unit-cell, is related to the Cabibo, Kobayashi, Masakawa time-reversal breaking in elementary particle physics.

Discovery of the time-Reversal odd Phase: These phases for various technical reasons are very hard to detect even when the magnitude of the order parameter is considerable ($O(0.2)\mu_B$ per unit-cell). However, some really fine experimentalists persisted in looking for it. Evidence for the broken symmetry came from dichroic ARPES experiments in BISCCO[7] but they were truly established by comprehensive

polarized neutron scattering, first in YBaCuO[5] and then in HgBaCuO.[6] Philippe Bourges has informed me that there is also evidence now for such a phase in LSCO. The line of phase transitions $T_p(x)$ in every case is similar to that in Fig. (1); its termination to the left of the phase diagram is still unknown. There are also features discovered in the experiments which are not predicted by the simple two-dimensional model through which they were motivated. But the basic symmetries are the same. At the very least, these discoveries rule out the Hubbard model as an adequate description for the Cuprates.

Thermodynamics at the Transition: It was soon realized that the classical statistical mechanical model for the observed phase is the Ashkin-Teller model which has four degrees of freedom per unit-cell corresponding to the four possible flux configurations predicted in the model. The Ashkin-Teller model, in the relevant range of parameters, has a line of Gaussian critical points. In this range, it has a rather smooth looking specific heat[12] resolving a major psychological problem for accepting that $T_p(x)$ is really a line of phase transitions. A weak singularity is expected in the uniform magnetic susceptibility at $T_p(x)$; this has now been found.[13]

Quantum-Critical Fluctuations: The quantum-mechanical generalization of the AT model including inertia and dissipation has the same properties in the quantum-critical regime as the dissipative xy model. The quantum critical properties depend crucially in the form of the dissipation. There have been speculations[9] that with the Caldeira-Leggett form of dissipation, the quantum-critical fluctuations may be local. Vivek Aji and I have succeded[10] in finding an asymptotic solution to this model. The critical properties are determined by the correlation in time of quantum-flips which change the local flux configurations. The quantum-critical fluctuations are precisely of the form which was hypothesized long ago to explain the properties of the strange or marginal fermi-liquid phase. There are aspects of these calculations which are being checked through quantum Monte-Carlo calculations by Asle Sudbo and others.

Superconductivity: Given the knowledge of the nature of the quantum critical fluctuations, their coupling to fermions could be found. The fluctuations are the correlations of the flips between the four possible local flux configurations of the AT model or in the xy model, changes in the local angular momentum of the collective current variables. They couple to local angular momentum of the fermions:[11] $\psi^+(\mathbf{r}\times\mathbf{p})\psi$. This produces, on integration over the fluctuations a four-fermion coupling which can easily be seen to lead to pairing in the d-wave channel. The same coupling and the same fluctuations determine the normal state self-energies so that the parameters can be read off from the ARPES experiments in the normal phase to show that estimates of T_c and the superconducting gap Δ do not rule out the mechanism. Detailed experimental tests of the theory are underway.

Pseudogap: An incomplete part of the theory of the Cuprates from this point of view is the nature of the single-particle spectra in the pseudogap phase below $T^*(x)$. It was suggested that this is a phase in which the fermi-surface gives way to four fermi-points.[3] Evidence for this has been found.[4] This may turn out to be

the most remarkable aspect of the revolutionary physics of the Cuprates because it violates[8] Bloch's theorem on gaps in spectra in condensed matter physics. One way to get around Bloch's theorem is for the fermions to have an effective infinite range interaction. Such effective interactions are indeed found in the vicinity of and just below T_p. The question, how the collective fluctuations in the time-reversal odd phase may generate such an effective interaction down to the lowest temperatures, is being pursued.

The essentials of the Physics of the Cuprates will have been deciphered if this last remaining issue is understood. Given the agreement of most of the properties of the Cuprates in the different regions of the phase diagram with calculations based on the quantum-critical point derived in the proposed model and the fluctuations of the time-reversal odd phase which has been discovered, it is to be expected that the spectra of the pseudogap phase is also a property of this phase.

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Discussion

M. Rice Ok, thank you very much Chandra. Any questions or discussions on this?

A. Georges If you force me to say something, I am going to ask Chandra a question, which he knows what it is: what is the reconstruction of the fermionic spectrum when these currents form? I think this is really an issue there because that is what decides whether the formation of the currents, which I think are there in the experiments, are important to understand the pseudogap state in the end.

C. Varma Yes, as you noticed, I have very carefully confined myself to discussing this region and the pairing mechanism here. The reason for that is that I do not have a theory that I believe about what is going on in this regime. Let me say that I think I begin to know what the interesting issues there are and they are very interesting problems in the sense that if you take an ordered pattern

that corresponds to a periodic vector potential, it looks completely benign. But imagine you make a quantum fluctuation where you flip one of these, now this object contains an arbitrary flux quantum which is the order parameter and now imagine you are calculating in that fluctuations spectra what is happening to the fermions going around and you see that the fermions cannot obey the periodicity of the lattice with an arbitrary flux locally here. So this becomes a very interesting lattice gauge theory problem that we are trying to solve.

A. Georges Maybe I can just add one thing, which is that as a theoretical question I think how these currents are generated in concrete model is still an issue: the single layer three band model I think has now been shown not to display the currents and apical oxygens must be included which may be actually consistent with the experimental observation that the moments are canted. So I would like to know...

C. Varma I would say that when you take the more elaborate model the symmetry of the pattern that is found is the same symmetry as in the simple model I had investigated. The symmetry elements do not change. So the statistical mechanics that I am talking about does not change.

M. Rice Thank you very much Chandra, I think we should move on and I would like to move on to a discussion about a topic that was not brought up in Catherine's talk: the recent experiments on the nature of the stripe phase and superconducting stripe phase. I have asked Phuan if he would just give a very short presentation on that and then Steve White has done some calculations relevant to that which he will tell us about.

Prepared Comment by Phuan Ong: Phase Coherence, Vortex Liquid and the Nernst Effect in Cuprate Superconductors

1. Phase rigidity of condensate

In the low-T_c superconductors, the wave function amplitude $|\Psi| \exp i\theta$ vanishes at the transition temperature T_c. As soon as the condensate appears at T_c, $\Psi(\mathbf{r})$ has the same phase θ everywhere (if the field H=0), i.e. long-range phase coherence prevails.

In the cuprates, however, there is now compelling evidence that a very different scenario occurs at T_c. We recall that, in the Kosterlitz-Thouless (KT) transition in two-dimensional (2D) systems, loss of phase coherence occurs at a temperature T_{KT} lower than the temperature at which $|\Psi|$ vanishes. At T_{KT}, the collapse of phase coherence results from the spontaneous unbinding of vortex-antivortex pairs driven by entropy gain. The 2D superconductor becomes unstable to the spontaneous appearance of *mobile* vortices at T_{KT}. Above T_{KT}, $|\Psi(\mathbf{r})|$ remains finite but the rapid diffusion of (anti)vortices leads to strong (singular) fluctuations in $\theta(\mathbf{r})$. More generally, 3D superconductors with highly anisotropic coupling, low superfluid density

ρ_s and large pair-binding energy may suffer a similar loss of phase coherence by vortex(loop) creation. We will call such transitions the phase-disordering scenario.

In 1995, Emery and Kivelson[1] pointed out that cuprates differ from low-T_c superconductors in having an anomalously low phase-disordering temperature. Terahertz experiments by Corson *et al.*[2] revealed that the kinetic inductance survives to 25 K above T_c in ultrathin films of Bi 2212.

Here, I wish to summarize recent experimental progress in establishing the phase disordering scenario and the high-field phase diagram of hole-doped cuprates. The abbreviations Bi 2212, Bi 2201, YBCO and LSCO refer to $Bi_2Sr_2CaCu_2O_8$, $Bi_2Sr_{2-y}La_yCuO_6$, $YBa_2Cu_3O_{6+y}$ and $La_{2-x}Sr_xCuO_4$, respectively.

2. Nernst Effect

In a Nernst experiment, the sample is subjected to a temperature gradient $-\nabla T \| \hat{\mathbf{x}}$ in the presence of field $\mathbf{H} \| \mathbf{z}$. In the vortex-liquid state, the gradient drives vortices with average velocity $\mathbf{v} \| (-\nabla T)$. Their motion produces an electric-field $\mathbf{E}_N = \mathbf{B} \times \mathbf{v} \| \hat{\mathbf{y}}$, which is detected as the Nernst signal $e_N = E_N/|\nabla T|$. In 2000, Xu *et al.*[3] reported the surprising finding that e_N, measured in LSCO, persists to an "onset" temperature T_{onset} much higher than T_c. Follow-up experiments[4] showed that, in LSCO, the curve of T_{onset} vs. hole doping x is dome-shaped just like the T_c-x curve, except that T_{onset} attains a peak at $x \sim 0.1$. If e_N arises from phase-slippage associated with vortex motion above T_c, the experiments imply that vorticity persists high above the T_c curve in the cuprate phase diagram. Similar results were obtained in single-layer Bi 2201.[4] This conclusion implies that i) the pair condensate survives above T_c, and ii) T_c corresponds to the loss of long-range phase coherence rather than the vanishing of the pair amplitude. Because T_{onset} penetrates high into the pseudogap state, Cooper pairing with vanishing phase stiffness must comprise a very significant fraction of the many-body state that defines the pseudogap.

Subsequent experiments[9] have investigated in detail the Nernst signal behavior above T_c in the hole-doped cuprates YBCO, Bi 2212, Bi 2223 (by contrast, e_N vanishes above T_c in the electron-doped cuprate NCCO). These experiments revealed that the e_N vs. H curves have the same general pattern across the hole-doped cuprates. At low T ($\ll T_c$), e_N is initially zero when $H \leq H_m$ (the melting field of the vortex solid). At H_m, the vortex solid melts, and e_N rises steeply to attain a broad maximum at $H \sim$15-30 T (depending on doping) in the vortex-liquid state. Beyond this peak, e_N falls gradually, reaching zero at a field that is identified as the upper critical field H_{c2}.[5,6,9] This characteristic "tilted hill" profile is observed in all the hole-doped cuprates. In the single-layer cuprates LSCO and Bi 2201, the maximum value of H_{c2} is estimated to be 80-90 T by extrapolation. However, in the bilayer cuprates Bi 2212 and YBCO, H_{c2} is much higher (at the scale of 130-200 T).

Strikingly, when T exceeds T_c, the same profile is observed, except that the peak value decreases rapidly with T. The vortex liquid extends to very high fields above

T_c. Interestingly, the estimated H_{c2} does not decrease to zero when T crosses T_c.[5,6] These results show quite directly that the vortex liquid extends smoothly to T high above T_c. At finite fields ($H > 1$ T, say), the magnitude of $e_N(T, H)$ varies very smoothly across T_c. The vortex liquid extends high into the pseudogap state above T_c.

These observations are in striking agreement with the phase-disordering scenario, but qualitatively incompatible with the BCS amplitude-closing scenario. A more detailed discussion appears in Wang *et al.*[9]

3. Torque Magnetometry

A consequence of the persistence of the vortex liquid high above T_c is that there should exist a large diamagnetism above T_c. In contrast with the very weak fluctuation diamagnetism that arises from Gaussian fluctuations in low-T_c superconductors, diamagnetism in the vortex liquid arises from singular phase fluctuations of the pair-condensate wave function. Hence it should be large, robust to intense fields (on the scale of H_{c2}), and strongly T dependent.

Because the diamagnetic currents are 2D, they exert a torque on the crystal in a tilted **H** which may be detected using high-resolution torque magnetometry. Extensive torque experiments have succeeded in resolving this unusual diamagnetism in Bi 2212[7,8] and LSCO.[10] On cooling from 300 K, the torque signal is initially dominated by the anistropic, paramagnetic Van Vleck term. Below T_{onset}, however, a diamagnetic contribution $M(T, H)$ grows rapidly to overwhelm the Van Vleck term. Both the T and H dependences of $M(T, H)$ match the corresponding e_N-T and e_N-H curves. The detection of this unusual diamagnetism provides firm thermodynamic confirmation of the vortex liquid state uncovered by the Nernst experiments.

Torque magnetization was also used to investigate in detail the divergence of $M(T, H)$ near T_c in Bi 2212. From measurements spanning 5 decades in H (5 Oe to 25 T), Li *et al.*[8] show that M is nonlinear in H in a broad interval of T above T_c. Expressing $M \sim H^{1/\delta}$, they found that the exponent δ (in weak H) diverges from 1.0 above 115 K to a value >6 as $T \to T_c$. They call the observed tendency of $\chi = M/H$ to diverge as $H \to 0$ "fragile London rigidity". High above T_c the vortex liquid displays long-range phase coherence, but the long-range stiffness is easily destroyed by fields of 10-100 Oe. Recent measurements on UD Bi 2201 and UD YBCO have uncovered the same fragile phase stiffness. Persistence of the gap in Bi 2212 to T even higher than T_{onset} has been observed by Yazdani's group using detailed scanning tunneling microscopy.[11]

4. Phase Diagram

Recently, torque magnetometry has been used to investigate the very lightly doped regime in LSCO in fields up to 42 T.[10] In Fig. 1, we summarize the torque and Nernst results in the 3D phase diagram with axes x, T and H. The T_c dome (in dark shade) is the region in which the vortex solid exists, and long-range phase coherence

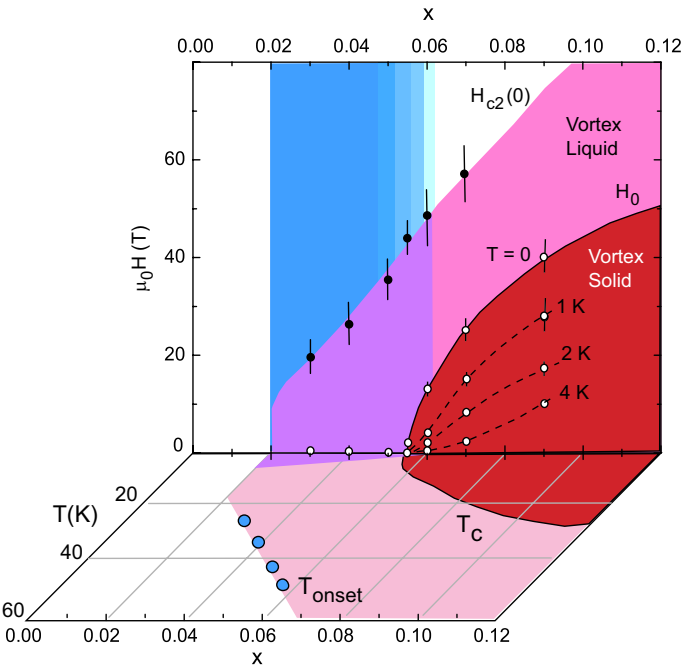

Fig. 1. 3D Phase diagram of LSCO with axes x, T and H inferred from torque and Nernst experiments. The vortex liquid surrounds entirely the vortex solid region (see text) [after Li *et al.*[10]].

prevails. Encompassing this dome is the vortex-liquid region (light shade). In the x-T plane (floor), this region is bounded by T_{onset}, while in the x-H plane it is bounded by $H_{c2}(0)$. Remarkably, for $x < x_c$ ($= 0.055$), the vortex liquid signal persists to H ~10-30 T even though the samples show no Meissner effect above 1 K.[10] This provides direct evidence that the pairing strength is extremely strong in the lightly doped samples even for $x = 0.03$. At these low hole densities, the ground state is a pair condensate with very weak phase stiffness. However, the condensate survives as a vortex liquid to intense H. The robustness of the pairing even for $x \ll x_c$ seems to be a key feature of the superconducting mechanism in the cuprates.

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Discussion

M. Rice Any questions or comments?

A. Stern Can we hear from Seamus Davis about the quasiparticle interference...

M. Rice Yes, Seamus will talk in a few minutes.

A. Auerbach There has been a lot of confusion in my mind at least on the sign of the magnitude of the Hall coefficient just below Tc. Can you clear up the situation or give any theoretical explication?

N. P. Ong Unfortunately, I did not prepare a slide. The first thing to note is the value of the upper critical field. The pair condensate survives to very, very high fields – 60 Tesla in my opinion is only half way there. We have direct evidence from the Nernst effect that, in this doping range in YBCO, the Nernst signal is just beginning to peak. So it is going roll down to zero in fields high above 60 Tesla. In all the hole doped cuprates, to my knowledge, the moment vortex flow begins, it pulls the Hall effect signal negative. In optimally doped YBCO, the change of sign is very abrupt. In underdoped YBCO, the onset of flux flow makes the Hall signal change sign at temperatures slightly higher than Tc, because the condensate really exists above Tc, as indicated by the Nernst signal, and confirmed recently by high-resolution torque magnetometry (which detects a sizeable diamagnetic signal extending above Tc). By measuring the resistivity alone above Tc, we might gain the (incorrect) impression that we are in the normal state. However, there already exists a pair-condensate. You can see it clearly in the Hall signal and Nernst effect. The vortex contribution to the Hall signal pulls the Hall sign negative above Tc. I believe that this effect has been ignored by Taillefer's group. Hence, given that the Hall signal from the flux flow effect is negative, one has to be cautious about identifying the sign of the pockets seen in the oscillations as electron-like. I think that the oscillations do correspond to a pocket, my quibble is whether the sign is negative.

A. Auerbach This negative Hall sign is in all materials or just YBCO?

M. Rice I would make a comment about that. The negative Hall sign is seen in the stripe phases already in the LTO and LTT phase. The Hall R_H drops dramatically, it is constant and hole- like above and drops dramatically and becomes negative at a lower temperature. In the samples where they have measured at high fields and seen the quantum oscillations, there is a similar temperature dependence, again there is an abrupt drop and it comes down and turns negative. So I think to my mind there is a real possibility that in these materials there is a superlattice of some form maybe similar to that in the stripes, forming at a higher temperature.

N.P. Ong I gather from your comments that you agree that the Hall sign is actually an electron pocket?

M. Rice That would be a possible interpretation.

N.P. Ong I guess I would again be cautious about that, because you can see that the onset of the change of sign corresponds to the onset of the Nernst effect.

A. Stern A question maybe of principle: this Nernst effect, I know there are various attempts to explain it simply within the more or less Aslamazov-Larkin fluctuations theory and say that there is a new seed... this large signal is simply because of the metal there is a very small Nernst signal. Is this large Nernst signal necessarily a signature of physics that is fundamentally different from the Aslamazov-Larkin fluctuations or can it be accommodated within the conventional theory of superconducting fluctuations? What is the status of the theory of this?

N.P. Ong If I understand from what you are saying, you are asking whether the Nernst signal can be identified with vortex flow?

A. Stern Whether it can be explained simply within conventional superconducting fluctuations theory.

NP Ong Yes, of course. In low T_c you will see this Nernst signal as well. However invariably it vanishes at T_c, the Nernst signal. It has been studied in the sixties...

A. Stern You are saying below T_c?

N.P. Ong Yes, so the Nernst signal appears below T_c in niobium and vanadium

A. Stern But there is also above T_c, enhanced signal in niobium silicon or something...

N.P. Ong Those measurements are in very thin films that are in the Kosterlitz-Thouless regime. That is the data of Behnia and originally the interpretation was that this was Gaussian fluctuations, even though in our opinion a Gaussian fluctuation is just too weak to generate such a huge Nernst signal. Subsequently they re-interpreted their results as arising from Kosterlitz-Thouless transition in thin films. So within Kosterlitz-Thouless systems, you will see a Nernst signal because it is a vortex liquid, above the K-T transition.

A. Stern But there is no way to explain this Nernst signal without vortices, just by fluctuations?

N.P. Ong Well, that is a separate question, so how do we know that it is due to vortices? The magnetization confirms that – i.e. the sign of the diamagnetic

signal.

C. Varma I was going to say that it is very easy to get large Nernst signal for instance in graphene: there is a gigantic graphene signal and only this non-linear magnetization together with the Nernst signal, which suggests that has to do with the Kosterlitz-Thouless type of physics. Also looking at the nice calculations by David Huse and Ashvin Vishwanath and others, what you discover is that you can quantitatively fit with the theory provided you assume that the vortex core energy is anomalously large. So what effectively they are saying is that you can understand the observations by having nothing to do much with cuprates physics specifically, except that you would see it in any Kosterlitz-Thouless type systems upon increasing the vortex core stiffness by a factor for about 4.

N.P. Ong Well, I am happy to talk to you later about how well the Huse and Sondhi calculations fit our data. But I think we have made a systematic study of the Nernst signal across the phase diagram, and as I showed in the sketch, it coincides with the superconducting dome. So, we have searched for the Nernst effect signals in many systems. In graphene, where we have seen it, it is only large at the Dirac point. So it is not that easy to see a large Nernst signal.

C. Varma Depending on the density of states near the chemical potential, you get a larger signal ... in kinetic theory...

NP Ong The Dirac point is very special. But, anyway, the magnetization data really clinch the issue for me.

B. Halperin I just wanted to say that there is a difference between qualitative and quantitative But in response to Ady's questions I think it would be fair to say that the Aslamazov-Larkin type of fluctuation, which is the basis of this Huse *et al.* paper, goes, in some sense, over into vortex motion, if one makes it rather 2 dimensional and you take the nonlinear terms into account. You gradually go over from something which is more Aslamazov-Larkin-like, which basically ignores the nonlinear terms, into a non-linear picture, and obviously it becomes quantitative. But I think qualitatively there is no sharp line between Aslamazov-Larkin fluctuations and the type of fluctuations, which you are using to explain things.

A. Stern But it makes sense to take these kinds of theories and calculate at 4 times T_c, temperature that it is so high ...

B. Halperin I think it depends on how large the gap is, and how strong the coupling is and so forth, but again it is a question of whether you talk about it quantitatively or qualitatively...

E. Shimshoni I just wanted to clarify on this point: the relation of the magnetization to the Nernst effect is it or is it not a smoking gun, so to speak, that it is a vortex physics and not Gaussian fluctuations?

N. P. Ong Experimentally we find that there is this interesting scaling between the Nernst magnitude and diamagnetic magnetization. So when you plot the two together both as a function of the field and temperature they are simply

proportional to each other. And I am not aware of any theory that links what is putatively a transport quantity to magnetization, although numerical simulations carried out by (Ashvin) Vishwanath and collaborators also see that in the 2-D Kosterlitz-Thouless system. It is an experimental finding that seems to invite further theoretical analysis.

M Rice Thank you Phuan, maybe Steve could give just a short summary of his recent calculations on this strong fluctuation stripe phase.

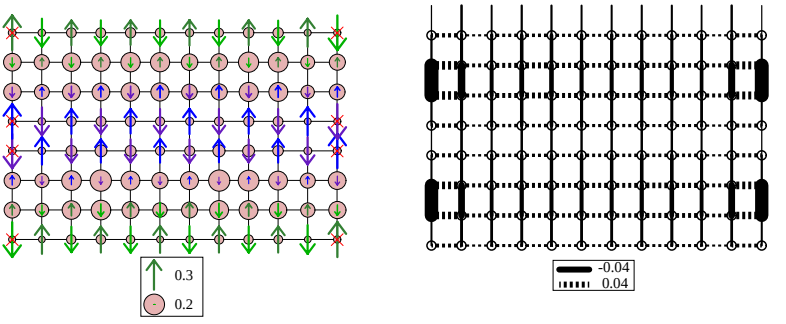
Prepared Comment by Steve White[a]: Pairing versus Stripes in the t-J Model

DMRG studies on the t-t'-J model in 2D have previously found striped states for small t', and weak or nonexistent pairing. For positive t', as $|t'|$ increases the stripes disappear and strong pairing occurs. For negative t', as $|t'|$ increases the stripes disappear but there is no pairing. We have recently found that for somewhat larger J values that strong pairing and stripes can coexist. Thus, we find that all four combinations of pairing/no pairing and stripes/no stripes occur in the t-t'-J phase diagram. In the pairing/stripes phase, we find only in-phase Josephson coupling between stripes.

Since the late 1990's, we have used DMRG methods[1] to study the ground state of first the t-J model and subsequently the t-t'-J model in two dimensions.[2] For the t-J model, we found striped ground states, and little signs of long-range pairing. The addition of a significant t' was found to destroy stripes, with $t' > 0$ favoring pairing and $t' < 0$ suppressing it. Recently we have found that pairing and stripes can coexist, and we have been able to study this interesting state.[3]

Our previous studies did not find ground states with both extended pairing correlations and stripes for two reasons: first, it has been difficult to construct limited-size clusters allowing significant particle number fluctuations on a stripe, and second, the model parameters which strongly favor pairing (e.g. $J/t \sim 0.5$, $t'/t \sim 0.2$) are different from the values usually taken to represent the cuprates (e.g. $J/t \sim 0.3$, $t'/t = -0.2$). In Fig. 1, we show results for a cluster which does have both stripes and pairing for $J/t \approx 0.5$, $t'/t = 0.0$. In order to allow hole fluctuations, a slightly anisotropic exchange interaction ($J_x = 0.55$, $J_y = 0.45$) was chosen to favor orienting the stripes along the x-direction, overcoming an opposite tendency due to the cylindrical geometry. Then, in addition to the magnetic fields at the open left and right ends, a pair field coupling has been applied to the ends of the stripes. The stripes persist into the central region of the cluster, where there are no applied

[a]This text has been coauthored by D. J. Scalapino.



(a) (b)

Fig. 1. (a) Hole $\langle 1 - n_i \rangle$ and spin densities $\langle S_i^z \rangle$ for a 16×8 lattice with $J_x = 0.55$, $J_y = 0.45$ and $t' = 0$, with cylindrical boundary conditions: periodic in the y-direction, open in the x-direction. A staggered magnetic field and pairing fields have been applied on the left and right edges to pin the stripes. A chemical potential $\mu = 1.23$ was used to give a doping of $x = 0.127$ in this grand-canonical simulation. (b) The pair field strength $\langle D_{ij} \rangle$ on each link for the system shown in (a).

fields, and so does the pairing. The results imply that static stripes and pairing, while driven by similar microscopic physics, can occur independently of each other or together.

In underdoped $La_{2-x}Sr_xCuO_4$, superconductivity and static spin density waves coexist.[4] Far-infrared measurements[5] find that the Josephson plasma resonance is quenched by a modest magnetic field applied parallel to the c-axis. Such a magnetic field is known to stabilize a magnetically ordered state[4] near $x = 1/8$, which is believed to be striped. Schafgans *et al.*[5] have argued that the establishment of the antiferromagnetic stripes leads to a suppression of the interlayer Josephson coupling. To explain this suppression, it has been suggested that anti-phase domain walls in the d-wave order paramater, locked to the SDW stripes, are stablized in the striped magnetic state,[6,7] and that the 90° rotation of the stripe order between adjacent planes leads to the cancellation of the interlayer Josephson coupling. Recent variational Monte Carlo[7] and renormalized mean field theory treatments[8] studied the possibility of anti-phase d-wave striped states.

In the striped/paired phase of the t-t'-J model, we have examined the possibility of anti-phase domain walls in the pair fields between stripes.[3] We found only in-phase order. By forcing the anti-phase order, we could measure a small energy penalty of order $0.01t$ per unit length for the anti-phase state.

We acknowledge the support of the NSF through grant DMR-0605444.

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Discussion

M. Rice Any questions or comments on this issue?

S. Sachdev I guess I am a bit lost on this. So there seems to be a real debate on whether you have an anti-phase domain wall. But what is the smoking gun experiment that you are trying to explain by this anti-phase domain walls?

M. Rice This drop in resistivity that Phuan Ong mentioned, it was a very sharp drop, and below this, the I-V curves are non-linear. And there are even signs of a Kosterlitz-Thouless transition at 16 Kelvin. This drop in the resistivity coincides, in the zero field, with the onset of the spin density wave magnetic signal. So that is why the belief put forward by Steve Kivelson and collaborators, was that this superconducting phase, that occurs here, had no Josephson coupling. Because even for very small Josephson interlayer coupling should one go below the Kosterlitz-Thouless transition, it should immediately lead you to full 3D superconductivity and full 3D superconductivity is only seen at 4 Kelvin. So this is the experiment that one is trying to explain. Chandra?

C. Varma I have a question based on my bemusement over the years about trying to investigate stripes. Is there anybody who believes that strips, that is that kind of order in the ground state, is a universal phenomena in the cuprates or, when it occurs, whether it has anything to do with anything other than itself? If it is not universal, why has it so much....

M. Rice I would say that the answer to your first question is no: I do not think the issue has ever been that it is universal. But to the second part of your question, I think it is relevant because if this phase, this idea of Steve Kivelson, is correct, that there is this set of antiphase boundaries in the superconductivity coexisting with domain walls in the spin density wave, that this is a stable phase in a certain doping range, then that shows there is a very intimate relationship between the magnetism and the superconductivity. They are not simply total competitors for the Fermi surface as you might have imagined. There is a very intimate sort of relationship between these two, which you find in other systems like two-leg ladders and systems like that.

C. Varma But does not your answer to my first question indicate that this is not one of the first ten things we should be worrying about in the cuprates? If it is not....

M. Rice No, to me it is a fascinating phenomenon and it may be a side effect, I do not want to argue that it is not, but I think it is a fascinating phenomenon. Any more questions? Subir?

S. Sachdev Well if I can follow Chandra's question. So I guess to some extent it means how we define stripes but if you define it as some sort of incommensurate spin or charge density with fluctuations, and if you include the STM measurements of Yazdani and Davis and neutron scattering measurements in the field, there are indications of these in almost all of the cuprates.

M. Rice These are not fluctuations. This is an ordered phase: the magnetic order is firmly established by neutron scattering in the absence of an external magnetic field.

S. Sachdev Absolutely here, but I was referring to Chandra's question more generally on all the cuprates. There are certainly indications of fluctuations.

A. Auerbach So you find it at doping 1/8 th, right? But would you find it at other dopings?

S. White Right, so at higher doping it, is a little bit harder to control the calculations but we do see it over a range, not just at 1/8 th. The higher doping regime we cannot say as much about it.

A. Auerbach But is it a commensurability effect?

S. White Ah, well ok... you can vary the doping by varying the spacing between the stripes and varying the filling along a stripe and it looks like that the filling along the stripe it seems like the minimum in energy is around one hole per two lattice spacings which is exactly right for 1/8 th but it is just a broad minimum. And so you can have a range. And so you can have stripes that have a greater density and you can also have stripes which are spaced further apart. And so we can see both possibilities but distinguishing exactly how that works is a little difficult to resolve.

E. Demler So Steve, some of the theories of stripes of course present frustrated phase separation so I have two questions: One, I do not see any phase separation although people have discussed it extensively (first question). The second is, well it is quite connected to the first one, how sensitive are your stripe phases to the boundary conditions?

S. White Ok, those are questions that we have had to deal with quite a bit from the early days. So the idea of frustrated phase separation was suggested by Emery and Kivelson and we argued against that, we have always argued against that, because we see it without the long-range Coulomb interaction which is the thing that would frustrate the phase separation. And we also do not see true phase separation where you have a Fermi thermodynamic higher density phase and a low density phase. So we just see stripe formations, not true phase separations.

E. Demler So just to understand, you are saying that the $t - J$ model in your conclusions never phase separates no matter how big J is?

S. White Oh, if you have big enough J's, it will phase separate but that is quite large, certainly bigger than one, maybe around $J/t = 2$... But the first sign

of phase separation may be where you have stripes formation and two stripes can attract each other. And so even then it is not quite so a clearcut. You know it does not look like the assumed phase separated phase that had been talked about in the early days. Ok, your second question was about whether the boundary effects are key in this. And so this was a motivation for us to orient these clusters with the stripes running along the cylinder so that there was no hard walls that set up a density wave going in that you might mistake for stripes. And so these stripes ran along and so at least it sort of removes that key effect of hard walls forming a density wave that you misinterpret as stripes.

M. Rice Ok, thank you very much Steve. I think we should move on.

Prepared Comment by Zhi-Xun Shen: the Pseudogap Phenomena in High Temperature Superconductors

1. Abstract

I have discussed the current status of our understanding of the pseudogap phenomenon in cuprates. Using doping, temperature and momentum dependent data from angle-resolved photoemission spectroscopy, I argued that the pseudogap is not a simple extension of the superconducting gap.

High temperature superconductivity in cuprate oxides remains an outstanding problem for quantum theory of condensed matter. Unlike a conventional superconductor whose normal state is a Fermi liquid with a well-defined Fermi surface, the normal state of cuprate superconductors is characterized by a peculiar energy gap usually referred to as the pseudogap.[1] I gave a brief summary of the pseudogap phenomenology and its relationship with superconductivity based on our most recent angle-resolved photoemission spectroscopy (ARPES) experiments.[2] Following the spirit of the Solvay conference, I will only use the sketches to illustrate the findings. A reader is referred to the original paper for data and more detailed information.[3-6]

Figure 1 depicts the phase diagram of high temperature superconductors highlighting the pseudogap phenomena. In the overdoped regime (δ higher than the optimal doping where T_c reaches maximum), the material has a large Fermi surface sheet in the normal state and a simple d-wave superconducting gap opens below T_c. In the underdoped regime, the normal state Fermi surface is no longer complete, with only a Fermi arc remains. A section of the Fermi surface near the Brillouin zone corner has been gapped – a phenomenon now commonly referred as the pseudogap.[1]

The nature of this gap and its relationship with the superconducting gap is a

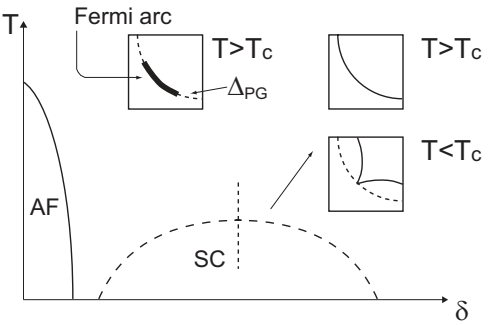

Fig. 1. Phase diagram of cuprates superconductors.

critical issue in the field now. In particular, the question remains whether the pseudogap is a simple extension of the d-wave superconducting gap to the normal state. I presented our recent doping, temperature and momentum dependent data suggesting that the pseudogap is not a simple extension of the d-wave superconducting gap. The physics is rather richer and more fascinating.

A. Doping Dependence [3-5]

The measurements at the lowest accessible temperature suggest the picture as in Figure 2. While the gap structure has a simple d-wave form for overdoped samples, it deviates from the form for underdoped samples. Further, the deviation gets pronounced in more underdoped samples. With δ changing, the nodal and antinodal regions behave very differently. The antinodal gap increases as the doping gets smaller, while near nodal gap (i.e., the gap along the Fermi arc) does not change much and, if any, deceases in deeply underdoped samples. As the doping approaching zero, the antinodal pseudogap smoothly connects to the "band structure" of the insulator – suggesting that the pseudogap is not a simple extension of the superconducting gap.

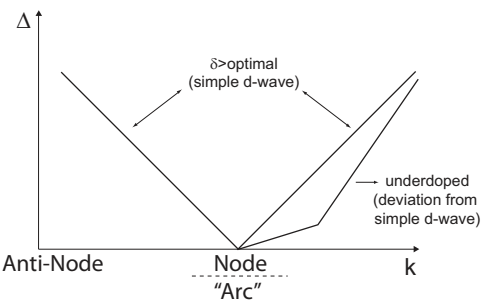

Fig. 2. Doping dependence of the energy gap form measured at low temperature.

B. Temperature Dependence [4]

At the superconducting transition temperature, a d-wave gap opens near the nodal arc region following a temperature dependence that resembles the opening of the BCS superconducting gap. This suggests that the gap on the Fermi arc in the underdoped samples is probably the true superconducting gap, with a caveat that more work is needed to check deeply underdoped samples. This behavior of the temperature dependence is very different for the antinodal region where the gap shows a small anomaly but does not close at T_c. These observations are summarized in the left panel of Figure 3.

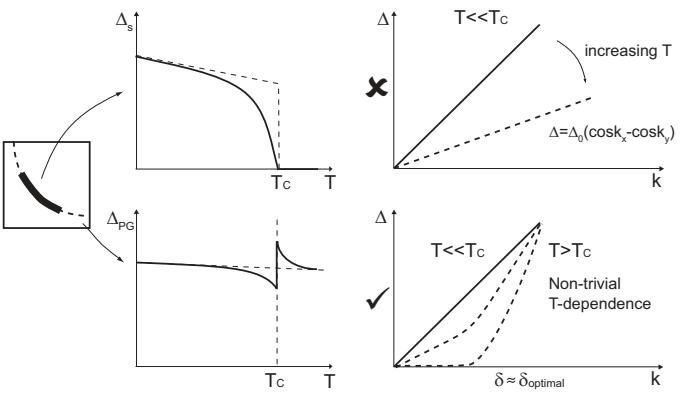

Fig. 3. Left: temperature dependence of the gap along the Fermi arc (upper panel) and at the antinodal region (lower panel). Right: the temperature dependence of the gap structure in momentum space, expected from simple d-wave (upper panel) and observed (lower panel).

An alternative way to look at the dichotomy of the temperature dependence of the energy gap can be illustrated in the right panel of Figure 3. If the gap is of simple d-wave form $\Delta = \Delta_0(\cos k_x a - \cos k_y a)$ whose momentum structure is independent of temperature, then it show keep its form as a function of temperature (upper right, Figure 3). Instead, the experiment suggests that it changes its momentum form with temperature that is far beyond what one expects from a trivial thermal effect (lower right, Figure 3).

The likely explanation for the unusual temperature dependence is that the pseudogap is not a simple extension of the superconducting gap but rather a distinct phenomenon. On the other hand, it is not totally unrelated either, especially approaching optimal doping where the distinction becomes less sharp. We note that gap structure of the nearly optimally doped samples measured at the lowest temperature is that of a pure d-wave form even though it has complex temperature dependence.

C. Momentum Dependence [6]

In a conventional metal, energy gap driven by Fermi surface instability (superconductivity and Fermi surface nesting driven density waves) occurs at the Fermi momentum k_f. As illustrated in Figure 4, the momentum location of the minimum energy gap in conventional materials is at k_f. In cuprate superconductors, recent data suggest that the momentum location of the minimum gap in the underdoped regime is not at k_f. This finding again suggests that we are not dealing with a conventional picture where a single superconducting gap opens along the underlying Fermi surface. A model with two energy scales can simulate the data.

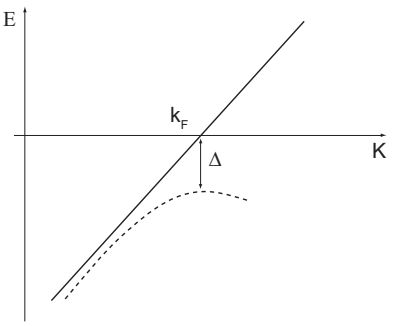

Fig. 4. Illustration of an energy gap opening expected in conventional superconductors.

In summary, the doping, temperature and momentum dependent photoemission data suggest that the pseudogap is a distinct phenomenon, not a simple extension of superconducting gap.

This work is supported by the US Department of Energy.

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Discussion

M. Rice Thank you very much Z.-X. Any questions or comments?

L. Glazman Could you elaborate a little more on these 2 pulse measurements? So you destroy the charge density wave locally or in which volume do you destroy the order?

Z.-X. Shen I would say we do not have a local probe, it is a macroscopic sample. On the other hand, if you really look at the system at the zeroth order, I have done so much violence to the system, will they still remember the momentum k? And it turns out that they do. All the things that I showed you, for example this observation of the amplitude mode, only happen very specifically in a small k-space location which is related to the normal state Fermi surface.

M. Rice Antoine, you wanted to say something?

A. Georges Yes, I wanted to come back on this issue that Z.-X. mentioned and which has been discussed quite a bit recently, that the slope at the gap near the nodes and the anti-nodal gap value might define two energy scales. So this emerged from Raman and photo-emission and also from STM and I would like to hear the opinion of people in the audience about that and also I would like to ask Z.-X. whether he believes, so this is this plot here, whether he now believes this is true down to very low temperature in the underdoped regime or this is essentially a finite temperature effect.

Z.-X. Shen Certainly, let us say there are certain material systems where T_c is very low. One of the things we have done, and it is also connected with the last two talks, is the stripe phase. We consider that really is the case that you have opportunity to probe at very low temperature while superconductivity is already gone. And that is more or less what I have traced, the gap structure in the strip phase, at 1/8 th superconductor. So I will be very happy to show the data. That is a very low temperature measurement.

A. Georges Ok, but how general is that? Now you mentioned strip compound...

Z.-X. Shen How general is that? From what we have measured we think it is very general As in this field there is a report on LSCO near optimum doping that they look different. That is still an experimental issue which remains to be resolved.

T. Senthil So this discovery that you have Fermi arcs in the experiments in the pseudo-gap regime is very fascinating. So motivated by that, one might want to ask the question whether the underlying ground state in this underdoped side is a metal which has disconnected Fermi arcs. You know where there are gapless electrons. Now, many people of course tried to describe this state over the last ten years and did not quite succeed, so I was wondering if anyone could think of a fundamental reason why such thing might be prohibited... either Fermi liquid or non-Fermi liquid ?

Z.-X. Shen I guess that is a serious question.

T. Senthil It is a serious question and I want to know if anyone had any comments...

A. Georges Well it seems to me that there are really two routes to this problem. What several people have tried to do is to come up with some genuine fixed point at zero temperature which would have basically nodal points at the Fermi surface. Another perhaps more conventional – that I think is still quite interesting – explanation is highly momentum selective incoherence of quasiparticles in the normal state. And I must say this is the picture which is also emerging from various calculations, either from the weak-intermediate coupling functional RG side or from the most strong coupling short-range DMFT side. And you know that Mott correlation basically destroys quasiparticles first at the anti-nodes.

E. Demler I can respond to Senthil's question. But there are also simple models, say that if you have fluctuating superconductivity that would give you this type of function you could also have srong Doppler shifts on quasiparticles, it would give you spectral functions that look precisely the same way... But I also have a question to Z.-X.: in the last few years, you were a strong advocate of the importance of electron-phonon coupling in the cuprates, so whether this picture of what you discussed fits into your understanding of electron-phonon interaction or not, I would be curious what you are thinking...

Z.-X. Shen Yes, there are two levels at issue. One is phenomenology, you see a kink around 50 mV as consequence of coupling to the optical phonons. I still believe so and I believe there is more and more evidence in support of that. What would this phonon do is another question and I believe strongly now there are very strong reasons to believe that as you go in the deeply underdoped regime, you see a very strong polaron effect. In very low doping and I think what demonstrates it very well is the long-term puzzle we had in this field ... For about ten years we asked this very basic question: what happens to the chemical potential when you dope an insulator? That is very basic, right? I dope an insulator, the chemical potential in any model in a trivial way should have either come to the conduction band or to the valence band. You have doping so much here. The chemical potential appears not to be shifting then it starts to shift in a very strange way, ok? And that has been a puzzle for me for about ten years and when you try to put that into the context of the polaron behaviour in this system, you can understand it in a very simple way. And that was demonstrated in the two papers by Zhou X. J. and his thesis.

X.-G. Wen This is a comment: there is a kind of theoretical prediction associated with the kink. And when there is a kink, then there is a possibility that actually there is a break, that is, there is a gap somewhere. So in a sense in a normal state you only have one band above zero (positive energy) but when you have a kink you may have several bands and there is a gap between different branches. So my point is that it is probably very worthwhile to measure this very carefully to see whether there is some kind of a gap away from zero energy. There is a

lot of rich structure there...

Z.-X. Shen Let me just acknowledge that my thinking on this issue and going towards that was actually stimulated with a discussion with Xiao-Gang quite a few years ago. This, I am heading towards directions there are two processes: one is there is a gap happening here and with arc left or could be a pocket which I cannot decide, then as you go into the superconducting transition there is a second superconducting gap open along this remnant of the Fermi surface. That is the direction we are going. Now, whether this is a gap of some sort of order or the system becomes so incoherent in accord with polaron-like behaviour, that is a question I cannot decide.

M. Rice So maybe we can move on and have a few words from Seamus.

S. Davis – prepared comment On "Cooper pair signature".

M. Rice Any questions or comments?

A. Georges Yes, I am happy to see some of these pictures seems consistent with what we have proposed based on the Raman, and I have a question which is what you call δ_c here what is the doping dependence of that?

S. Davis So that is optimal doping and this one is about 7 %, so that is a wide piece of the doping range. So we do not think that the magnitude of the superconducting energy gap is falling, we think it is rising but it is being cut off by the fact that the arc of coherent excitations is moving closer to the ($\pi/2$-$\pi/2$) point.

T. Senthil Seamus, the data that you showed and that you have talking about for a while, that the anti-nodal states are very incoherent so that you can not see any quasiparticle interference coming from them...

S. Davis Yes so may I respectfully interject something, I think they are perfectly coherent in real space, it is only in momentum space that they are incoherent.

T. Senthil Yes, in momentum space.

S. Davis Ok.

T. Senthil Yes, so does that kill the idea that has become very popular in the last year, that it is electron pockets that are deciding behaviour in those regions...

S. Davis Yes, I would need to do that experiment before I could attempt to answer that question. I mean of course the hidden problem with the electron pockets, there is another hidden problem with the electron pockets story: why is that the state that you generate at 60 Tesla may not be the normal state of the cuprates at zero Tesla. It may be another state which has yet to be identified.

M. Rice In the last presentation, Bernhard Keimer wants to make some remarks about more general ways of looking for new superconductors.

Prepared Comment by Bernd Keimer: Perspectives in Orbital Physics

1. Historical Note

The field of orbital physics was established more than fifty years ago, when Wollan and Koehler[1] used the newly developed technique of neutron scattering to elucidate the lattice and magnetic structures of $La_{2-x}Ca_xMnO_3$. In this material, each Mn ion is surrounded by an octahedron of six oxygen ions. The nearly cubic crystal field splits the degenerate manifold of atomic d-electrons on the Mn ions into a lower-lying triplet of t_{2g} symmetry and a high-energy doublet of e_g symmetry. Further distortions away from cubic symmetry also split the residual degeneracy of the sub-manifolds. For cations with incompletely filled d-electron shells, such as Mn^{3+} in $La_{2-x}Ca_xMnO_3$, electronic energy can thus be gained by a cooperative distortion of the octahedra through the Jahn-Teller effect, which sets in below the so-called "orbital ordering" temperature. The relative orientation of orbitals on neighboring cations then determines the superexchange interactions between the unpaired d-electrons, which in turn determine the magnetic ordering patterns. This was first pointed out by Goodenough in a paper[2] that immediately followed the one by Wollan and Koehler. Goodenough's model, together with concurrent work by Kanamori and Anderson, led to a qualitative understanding of the magnetic interactions and their dependence on orbital occupation, establishing a framework now known as "Goodenough-Kanamori-Anderson rules". This framework has been used extensively to interpret data on transition metal oxides up to the present day.

The description of the interplay between spin, orbital, and charge degrees of freedom resulting from this pioneering experimental and theoretical work was later refined and extended to many other materials. Following the discovery of high-temperature superconductivity in cuprates, a multitude of new materials were discovered, including many in which orbital degeneracy plays a central role.[3] While most of the work prior to 1986 was carried out on polycrystalline samples, a concerted effort led to the synthesis of high-quality single crystals and crystalline thin films of many transition metal oxides.

2. Some Trends in Current Research

Following these advances, it proved possible to experimentally probe the *dynamics* of spin, charge, and orbital degrees of freedom, and to quantitatively determine the energy scales of the corresponding excitations, using modern inelastic neutron and x-ray scattering as well as optical and photoemission methods. The excitation spectra of Mott insulators are an excellent testing ground for current theories of orbitally degenerate transition metal oxides, not only because of the theoretical simplification implied by the absence of low-energy charge excitations, but also because structurally simple model compounds without chemical disorder are available. While the

prototypical Mott insulator $LaMnO_3$ (whose valence electron resides in the e_g orbital manifold) appears to be amenable to a quantitative description in terms of superexchange models developed in the 1950's,[4] the adequacy of this framework for other Mott insulators with identical crystal structures is at least questionable. Discrepancies between theoretical predictions and experimental data are particularly striking in compounds with partially occupied t_{2g} orbitals, such as titanates and vanadates, where the orbital degeneracy is large and the coupling to the lattice relatively weak.[5–7]

Theoretical work employing an "orbital pseudospin" formalism first introduced by Kugel and Khomskii in the 1970's[8] indicates that qualitatively new behavior can emerge in Mott insulators when spin and orbital excitations have comparable energy scales, as appears to be the case for titanates and vanadates. In particular, it has been predicted that spin and orbital order can be obliterated completely, even in materials with three-dimensional bond networks, due to the large Hilbert space for combined spin-orbital fluctuations, giving rise to novel spin-orbital quantum liquids.[9] These models do not capture the full structural and chemical complexity of the real materials, neglecting for instance ionic size-driven lattice distortions and electronic orbitals centered on counter-anions in the perovskite structure. The influence of these features on the spin and orbital excitation excitation spectra is currently a subject of active investigation by ab-initio numerical calculations.[10,11] At the same time, thermodynamic evidence is accumulating in favor of spin-orbital liquid states both in the Mott-insulating titanates[12] and in compounds with geometrically frustrated magnetic interactions.[13] It thus appears likely that orbitally degenerate Mott insulators will be an interesting subject of research for many years to come.

The introduction of mobile charge carriers into compounds with nearly degenerate *d*-orbitals raises a number of additional interesting (and sometimes contested) issues, such as the existence of orbitally selective metal-insulator transitions[14] and electronic analogues of liquid-crystal mesophases[15] in ruthenium oxides. The spin-orbital polaron concept has been fruitful in attempts to explain the unusual magnetic, optical and transport properties of doped transition metal oxides with orbital degeneracy, including charge-ordered and metallic states in manganates[16] and the unconventional superconducting state in cobaltatates.[17] However, theories of these phenomena have not yet reached the same predictive power as those for the Mott-insulating systems. A particularly difficult issue is the influence of disorder, which is invariably associated with doping by chemical substitution. Experimental evidence indicates that disorder can have a decisive influence on the phase separation between different electronic phases in manganese oxides, and hence on the macroscopic transport properties including the "colossal" response to external electric or magnetic fields.[18] While this is obviously important for technological applications of transition metal oxides, the theoretical description of orbitally degenerate transition metal oxides in the presence of disorder is still in its infancy.[19]

3. Perspective: Oxide Heterostructures

Modern synthesis methods, based in particular on advanced pulsed-laser deposition technology, have recent yielded high-quality heterostructures of transition metal oxides with atomically sharp interfaces. Early examples include work on the macroscopic magnetization of ferrate[20] and manganate[21] heterojunctions. Because of the large variety of electronic phases present in bulk transition metal oxides, these developments open up entirely new perspectives for the exploration of correlation-driven interface phenomena, akin to the development of semiconductor physics some fifty years ago. A model system that has captured particular attention in the past three years is the interface between the two insulators $SrTiO_3$ and $LaAlO_3$. Interest in this system was stimulated by the discovery of a metallic state at the interface,[22] followed up by the observation of a two-dimensional superconducting state at low temperatures.[23] It was also found that the interface conductivity depends critically on the thickness of the $LaAlO_3$ layers and can be effectively controlled by a gate voltage in a field-effect-transistor arrangement when the layer thickness is close to the threshold for metallicity.[24] Research on the origin of this behavior has focused on the interplay between dipolar fields at the interface and at the surface,[25] the relaxation of the atomic positions in both constituent layers,[26] and the role of oxygen defects.[27] While the relative importance of these effects is still under investigation, lateral structuring techniques are now being explored vis-à-vis electronic device applications.[28] Electronic reconstructions akin to that of the $SrTiO_3$-$LaAlO_3$ system have also recently been explored in heterojunctions based on vanadates.[29]

Whereas bulk $SrTiO_3$ and $LaAlO_3$ are band insulators without electronically active transition metal *d*-orbitals, the control over interfacial states also offers tremendous opportunities for orbital physics. First, by virtue of either the "polar catastrophe"[25] or the work function mismatch between two oxides, charge carriers can be introduced near an interface without introducing chemical disorder. The charge transfer across oxide heterointerfaces is just beginning to be explored by interface-specific probes such as x-ray absorption,[30] photoemission,[31] and electron energy loss spectroscopies[32] as well as spectroscopic ellipsometry.[33] A second opportunity lies in the manipulation of the orbital occupation, either by using interfacial misfit strain to influence the ligand positions and hence the crystal fields acting on the transition metal ions,[34] or through covalent bonding across the interface.[30]

Because of the dependence of the superexchange interactions on the orbital occupation discussed above, it is thus conceivable to "engineer" interfaces with specific superexchange Hamiltonians. One of the exciting perspectives of this approach is the controlled synthesis of new superconductors. Specifically, it has been suggested[35] to re-create the $t-J$ Hamiltonian presumably responsible for high-temperature superconductivity using heterostructures of $R\mathrm{NiO}_3$ (where R is a trivalent rare earth ion) and band insulators such as $LaAlO_3$. The Ni^{3+} ions (electron configuration $3d^7$) adopt a low-spin configuration with spin 1/2, formally analogous to the Cu^{2+} ions in the high-temperature superconducting cuprates. The phase diagram of bulk RNiO3

includes charge-ordered antiferromagnetic as well as metallic phases, but no superconductivity. The main difference between the nickelates (and other spin 1/2 metal oxides based, for instance, on Ti and Co ions) on the one hand and the cuprates on the other hand is the degeneracy of the e_g orbitals originating from the high symmetry of the metal-oxide octahedron. While this degeneracy is completely lifted in the cuprates such that the conduction electrons occupy exclusively the Cu x^2-y^2 orbitals, the near-degeneracy of the Ni $3z^2$-r^2 and x^2-y^2 orbitals reduces the effect of electronic correlations (an essential prerequisite of high-temperature superconductivity) and promotes Fermi-liquid-like metallic states in the nickelates. The insulating layers in the proposed heterostructures serve the dual purpose of suppressing the hopping of conduction electrons in the direction perpendicular to the interfaces (resulting in a quasi-two-dimensional electronic structure) and stabilizing the x^2-y^2 orbital of Ni through tensile epitaxial strain. The result is a self-doped quasi-two-dimensional electron system with a low-energy Hamiltonian closely analogous to that of the high-temperature superconducting cuprates. Detailed calculations have shown that this approach should indeed generate a single, hole-like Fermi surface closely resembling that of the cuprates,[36] and that the superexchange parameter J (which presumably controls the superconducting transition temperature) may equal or even exceed the one in the cuprates.[35]

These developments may indicate the end of an era in which physicists had to rely solely on solid-state chemists to synthesize new materials with unpredictable crystal structures in order to explore ideas on many-body interactions. Heterostructures of oxides and other complex materials may offer a platform to explore these ideas (and to optimize functionalities such as superconductivity) in a more controlled and systematic way in the future.

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Discussion

M. Rice Thank you very much Bernhard for reminding all us theorists that high temperature superconductivity is a very subtle phenomenon. And we are still a long way from declaring victory in this... Well ok, one comment Marvin!

M. Cohen In defense of theory, and this is not a very humble statement, but you mentioned strontium titanate. I predicted superconductivity in strontium titanate and successfully predicted six other superconductors.

M. Cohen Cohen I apologize for making this "not very humble" remark, but sometimes theory and theorists need defending.

M. Rice Oh ok, good... thank you all!

Session 4

Quantum Hall Systems, and One-Dimensional Systems

Chair: *John Chalker*, University of Oxford, UK
Rapporteurs: *Ady Stern*, Weizmann Institute of Science, Israel and *Ian Affleck*, University of British Columbia, Canada
Scientific secretary: *Serge Massar* (Université Libre de Bruxelles)

Rapporteur Talk by A. Stern: The Quantum Hall Effect – an Overview

1. Abstract

After briefly reviewing the main paradigms of our theoretical understanding of the quantum Hall effect, I highlight some issues which I believe are not well understood. Included in those are the robustness of fractional charge at rather high temperature, the elusiveness of the universal limit for the physics of the edge, and the physics of non-abelian quantum Hall states. I conclude with a few comments on the role the physics of the quantum Hall effect plays in some other condensed matter systems.

2. Introduction

This paper is written for the proceedings of the Solvay conference on "Quantum Theory of Condensed Matter". Naturally, the history of the Solvay conferences cannot be disregarded when the paper is written. "Who knows" says the humble author to himself "the Solvay Institute may be sending copies of the proceedings to all those legendary speakers of the early twentieth century. What can I write that would have been of interest to them?" Well, the quantum Hall effect – which is

the subject assigned to me to report on – suggests several possible answers to this question. The choice I made is to start with a brief reminder of the basic paradigms on which our understanding of the effect is based, and a more detailed discussion of some intriguing questions that are not yet answered.[1]

Let us start with the main paradigms. Following three decades of extensive study, we believe that we understand the basic characteristics of the quantum Hall effect in its ideal limit, that of zero temperature and infinite samples: when a system is in a quantum Hall phase, bulk states at the chemical potential are localized. Delocalized states are confined to the edges, and form propagating chiral edge states. This is why current flows with no dissipation. The physics of the edge states is intimately connected to that of the bulk, and the low energy edge physics is universal. Finally, we believe that the fractional quantum Hall effect is characterized, as counter-intuitive as it initially sounds, by quasi-holes and quasi-particles that carry a quantized fractional charge and satisfy a quantum fractional statistics.

Many theoretical approaches have been useful in the construction of this picture. For our present context, one such tool – that of flux attachment – deserves a special mention. Roughly, this is flux attachment: we start from Schroedinger's equation, $H\Psi = E\Psi$, where H is the Hamiltonian of interacting electrons in a partially filled Landau level, with $1/\nu$ flux quanta per electron and a random disorder potential, and Ψ is the many body wave function. Flux attachment is the definition of a new wave function F that describes "composite particles" (charged particles carrying α flux quanta each) at an average magnetic field of $\frac{1}{\nu} - \alpha$. There are two ways to do that. The first, which is at the root of Laughlin's wave function and Jain's subsequent works, is the definition

$$\Psi(\{z_i\},\{z_i^*\}) = \prod_{i<j}(z_i - z_j)^\alpha F(\{z_i\},\{z_i^*\}) \tag{1}$$

while the second, which is at the root of composite fermion and composite boson field theories, replaces the Jastrow factor $\prod_{i<j}(z_i - z_j)^\alpha$ by a phase factor $\prod_{i<j}\frac{(z_i-z_j)^\alpha}{|z_i-z_j|^\alpha}$. The virtues and flaws of these two ways are beyond the scope of the present presentation. What is crucial at the present context, however, is the quantum statistics of the "composite particles", determined by the behavior of the function F under the interchange of two particles. The function Ψ is of course odd under such an interchange, being a fermionic wave function. The function F is then fermionic for even α, bosonic for odd α and anyonic for a fractional α. Flux attachment reveals how the fractional quantum Hall effect breaks the dichotomy between fermions and bosons, which is so prominent at zero field. The physical properties of fermions at $\nu = 1/3$ resembles that of bosons at $\nu = 1/2$ much more than that of fermions at $\nu = 1/2$, to give just one example[2]-.[4]

3. Abelian Quantum Hall States

The most prominently studied series of FQHE states, $\nu = n/(2n \pm 1)$ with a positive integer n, is best understood by the the choice $\alpha = 2$, that maps the FQHE onto an

integer quantum Hall effect for composite fermions, whose Landau filling factor is n.[2] This choice, known as "composite fermion theory", leads to several conclusions: first, at $\nu = n/(2n+1)$ a FQHE state is formed, with the bulk being gapped, and the gap decreasing with increasing n (in fact, composite fermion theory also allows for an exact analytical calculation of the gap in the limit of large n for the case of a clean system and an unscreened Coulomb interaction between the electrons). Second, for the series of $\nu = n/(2n+1)$ the edge is made of n co-propagating edge channels, whose low energy physics, in the presence of mutual interactions and inter-channel tunneling, is universal (for the $\nu = n/(2n-1)$ there is also a counter-propagating edge channel). Third, the charge of the quasi-particles/holes is $\pm e/(2n \pm 1)$, and finally, their statistical angle is $2\pi(1 - \frac{2}{2n\pm1})$.[5]

Real life samples do not follow the ideal assumptions of the calculations, and therefore observations differ from the ideal picture. The varying levels of robustness of different aspects of the picture is illuminating. Interestingly, the most robust aspect is the fractional charge. Second in line is the fractional quantum Hall effect itself, and last is the physics of the edge. The fractional statistics is a subtle issue, and hard to place in this line. Let us expand on these last few sentences.

The charge of a particle may be measured in several ways. For incompressible states, i.e., when the FQHE is observed, the charge may be measured through shot noise. For compressible states one may think of at least two ways for measuring the charge. One is through geometric resonances in the absorption of surface acoustic waves.[6,7] These resonances take place for particular values of the ratio of the wavelength of the acoustic waves to the cyclotron radius of the particle that absorbs the wave. The cyclotron radius is p/e^*B with p being the particle's momentum, e^* its charge and B the magnetic field. If we assume the momentum to be the Fermi momentum, i.e., the momentum dictated by the average distance between electrons, we may extract the particle's charge from the magnetic fields at which resonances are observed. If this charge is indeed $e^* = e/(2n+1)$ for filling factor $\nu = n/(2n+1)$ then the cyclotron radius that would be extracted from the resonances is $p/e|B - B_{1/2}|$, with $B_{1/2} = 2\Phi_0 n$ being the magnetic field that corresponds to $\nu = 1/2$.

Such resonances are experimentally observed by Willett and collaborators[8] for magnetic field values that correspond to rather large values of n, much larger than those where the FQHE is observed in the resistivity matrix. Furthermore, they are observed at rather high temperatures, larger than those at which the FQHE is observed.

We should pause to make a comment here. Within composite fermion theory, it is customary to think of the composite fermion as a particle of charge e that experiences a reduced magnetic field $\Delta B \equiv B - 2n\Phi_0$. The equivalence of this picture to the picture we present here, of a particle of charge e^* that experiences a magnetic field B, is a consequence of the identity $e^*B = e\Delta B$, and was discussed at length in the literature.[9] We do not dwell on it here, but limit ourselves to saying that the physical magnetic field in the system is obviously B and not ΔB.

Another way of measuring the charge of a particle is through the frequency of its

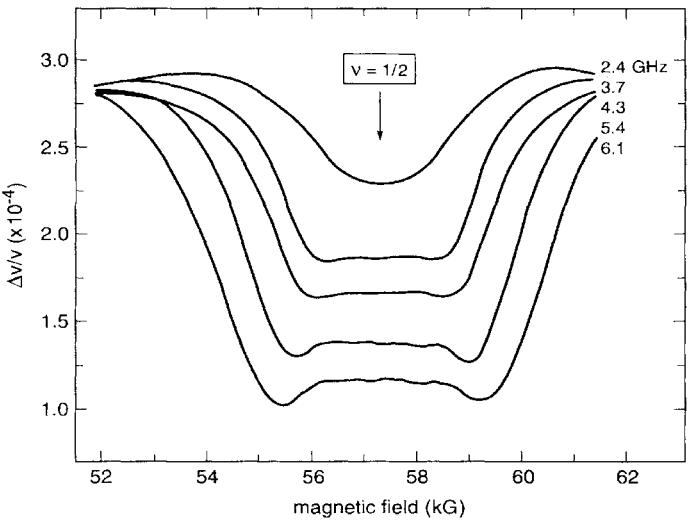

Fig. 1. Measurement of geometric resonances due to the interaction between a surface acoustic wave and a composite fermion that moves along a cyclotron orbit, by Willett et al. Resonances are observed for values of n as high as 13, beyond where the quantum Hall effect is observed. Reproduced from Ref. 8.

cyclotron resonance, taking place at $\omega = \frac{e^* B}{m^* c}$, where m^* is the particle's mass, and c is the speed of light. There are some subtleties involved in this measurement, however. An electronic system under a magnetic field must have an electronic cyclotron mode that approaches $\omega = eB/mc$ in the limit $q = 0$, as a consequence of galilean invariance (Kohn's theorem). This mode saturates the f-sum rule. In looking for a cyclotron mode for a quasi-particle, then, we look for a mode that has a subleading weight at the limit of small q. Furthermore, the mass of the particle need not be the bare mass of the electron, since its motion is controlled by electron-electron interaction. And even worse, it is not at all obvious that this mass does not depend on the frequency of the measurement. If this latter dependence is assumed to weak, and if the charge is again assumed to be $e^* = e/(2n+1)$, then the cyclotron frequency of the quasi-particles should be proportional to $B - B_{1/2}$.

This dependence is observed by Kukushkin, Smet and collaborators,[10] and observed at magnetic fields that correspond to values of n that are much larger than those for which the FQHE is observed, and at temperatures higher than those needed for the observation of the FQHE.

Both these measurements show the same point: the "composite fermion", an electron whose charge is reduced by the attachment of two flux quanta from e to $e(1 - \frac{h\sigma_{xy}}{e^2})$, is an entity that is reflected in observable quantities at temperatures and disorder at which the two dimensional electronic system is compressible, and no signature of the quantum Hall effect is observed. Put in other words – the physical process which we mathematically describe by flux attachment holds beyond the

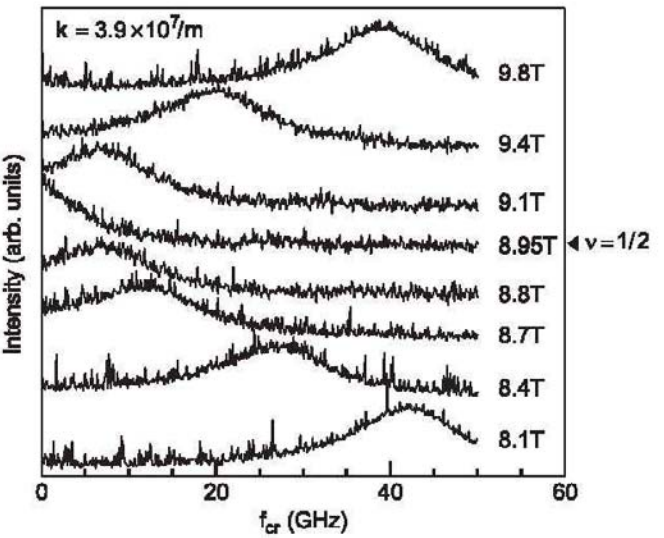

Fig. 2. Measurement of the cyclotron resonance of composite fermions at filling fractions close to 1/2, by Kukushkin et al. The resonances are observed at values of n as high as 25, at a temperature of 400mK. Reproduced from Ref. 10.

FQHE. In fact, the way in which it ceases to be valid, i.e., the way the two flux quanta are detached from the electron at high temperature is not described by present theories.

This is not only a high energy question: the attachment and detachment of flux is a process that may take place also at zero temperature: Consider a system of two identical layers at a Landau filling of $\nu = 1/2$ each (a total filling of $\nu = 1$). If the layers are far away from one another their coupling is weak, and the starting point to consider is obtained by attaching two flux quanta to each electron, in such a way that each electron interacts only with the flux tubes of electrons in its own layer. The two layers then form two weakly coupled Fermi liquids of composite fermions at zero average magnetic field. In contrast, if the layers are very close to one another, their coupling is strong, and the appropriate flux attachment is different: One flux quantum is attached to each electron, and each electron interacts with the flux tubes of electrons in both layers. The electrons are then transformed to composite bosons at zero average magnetic field.

Obviously, the physics of two coupled Fermi systems is very different from that of two coupled Bose systems. Examples to that difference are to be found in interlayer tunnelling and in the resistivity of the two-layer system to the flow of antisymmetric current. In the weak coupling limit, the slow relaxation of charge densities within each layer, driven by the small σ_{xx}, strongly suppresses tunnelling at low voltage,[11] while for strong coupling there is a Josephson-type peak of the tunnelling

at zero voltage.[12] Furthermore, at the weak coupling limit the resistivity to the flow of anti-symmetric current results (in the limit of a clean system) from inter-layer momentum transfer through electron-electron interaction,[13] while in the strong coupling limit such processes are blocked by the superfluidity of the two bosonic layers.[14] The predictions of these two choices of flux attachments are confirmed by experiments carried out in the two limits of weak and strong coupling. It is not at all clear, however, what happens in between these two limits. Is the transition between the two of first order? Can there be a continuous transition between two types of flux attachments? Are there any phases in between the two limits? Experiments indicate a co-existence of the two phases in the intermediate regime, which may be attesting for a first order phase transition. Numerical calculations raise also other possibilities, and theoretical considerations raise the possibility of intermediate states between the two limits. Altogether, the bi-layer $\nu = 1$ system is an example to the success of the idea of flux attachment, but also to the limited level of understanding we have of that idea.

Let us now turn from the bulk to the edge. Our understanding of the physics of the edge of a FQHE system is based on several key ingredients:[16] first, the edge of every quantized Hall bulk must be compressible. Second, composite fermion theory predicts a certain structure of the edge, including an enumeration of the edge channels and a determination of their low energy action. Third, certain parts of the low energy physics of the edge are expected to be universal. These include the heat capacity, the low energy response functions, the low energy $I - V$ characteristics of tunnelling of electron into an edge and that of inter-edge tunnelling of quasi-particles across a bulk quantized Hall state. The precise meaning of the term "low energy" is not obvious, however. There are several high energy scales in the problem, which in descending order are – the cyclotron energy $\hbar\omega_c$, the interaction energy $e^2/\epsilon l_H$ and the energy gap. Naively, one may expect the low energy limit to hold when the temperature and voltage are much smaller than these scales. Real life samples defy this expectation in several ways. Due to the presence of various screening gates, the edge velocities vary as a function of position, particularly close to quantum point contacts, which are the devices often used to test the low energy physics. Localized states within the bulk allow for energy dissipation from the edge to the bulk. The application of a voltage affects the bare amplitude for tunnelling across a point contact even when the voltage is low. And various conditions lead to a reconstruction of the edge structure. None of these invalidates the predictions of Luttinger liquid theory for the low energy physics, but each one of them makes the low energy limit harder to access.

Indeed, experiments on the structure of the edge are only at partial agreement with the theory. Among the many existing examples, we review here just one – the tunnelling of electrons from an external metallic tip to a quantum Hall edge. Luttinger liquid theory predicts a power law dependence of the current on voltage $I \propto V^{(1+\alpha)}$, with a power law α that depends on the bulk filling factor ν. Such a dependence is observed. However, the dependence of α on ν is not well understood,

both theoretically and experimentally.

The basic theoretical picture of tunnelling is always the same – the electron has to go under an energy barrier. The amplitude for tunnelling and the voltage dependence of the tunnelling current are determined by the action accumulated while the electron tunnels. Part of the energy that forms the barrier originates from the Coulomb interaction between the tunnelling electron and the electrons in the system to which it tunnels. The action accumulated due to this interaction depends on the interaction energy and on the time it takes the charge added to the system to disperse to its boundaries. Both the energy and the time depend on the filling fraction, and these dependencies determine the value of $\alpha(\nu)$. Theoretical analysis of $\alpha(\nu)$ resulted in two contradicting dependencies. A hydrodynamical model of the relaxation of the charge from the tunnelling point to the boundaries yields $\alpha = \nu^{-1} - 1$.[20] This model assumes that the relaxation of the charge involves only the charge mode on the edge. A different model tries to take into account the fermionic nature of the composite fermions, and finds that as a consequence of that nature, the relaxation process of the charge involves also dynamics of the neutral edge modes. This model finds $\alpha = \nu^{-1} - \nu_{cf}^{-1}$, with ν_{cf} being the composite fermion filling fraction.[19]

There is no clear experimental ruling between these two theories. Two experiments[17,18] have probed tunnelling into the edge, with one leaning to the first theory and the other to the second. The issue is still open for further exploration.

The characteristics of tunnelling into the edge of a fractional quantum Hall state is just one example to the difficulty in reconciling real life samples with the theoretical low energy limit obtained usually by means of a renormalization group analysis of the Luttinger liquid models for the edge.

4. Non-abelian Quantum Hall States

Presently, a large part of the research effort in the field of the quantum Hall effect is devoted to non-abelian quantum Hall states.[21] The defining properties of these states are theoretically understood: in the presence of localized quasi-particles that are kept at large distance from one another, the ground state of the system is degenerate, and the degeneracy is exponential in the number of quasi-particles. When the position of quasi-particles is braided adiabatically, a unitary transformation is applied to the system. This unitary transformation takes the system from one ground state to another. It depends (up to an abelian phase) only on the topology of the braiding. Furthermore, local perturbations to the system (phonons, electromagnetic radiation etc.) do not have matrix elements that couple different ground states. These properties make non-abelian quantum Hall states appealing candidates for the construction of topological quantum computers.

Many questions related to non-abelian quantum Hall systems are not yet answered. Among these, arguably the most burning one is – "do they actually exist?" In principle, one may imagine two ways to answer this question – numerically and ex-

perimentally. Numerical calculations give very strong support to the Moore-Read[22] proposition that the $\nu = 5/2$ FQHE state is non-abelian, and a rather strong support for the existence of a non-abelian $\nu = 12/5$ state. There is also strong support for the bosonic $\nu = k/2$ states being non-abelian, with k an integer going as high as 12.[21]

An experimental demonstration of non-abelian quantum Hall physics is a great challenge. As explained above, the characteristics of non-abelian physics are ground state degeneracy and the topological effect of braiding. A degeneracy, or almost degeneracy, of the ground state would be reflected in the heat capacity at very low temperatures, but the heat capacity, as well as other thermodynamic quantities, is a very hard quantity to measure.

Most of the "Litmus tests" proposed so far for non-abelian physics address the topological effect of braiding, and do so using intereferometry.[21] This is a natural choice since an interference experiment may examine the effect of particles encircling one another by having localized particles trapped in the loop defined by the two interfering partial waves. For non-abelian states, the outcome of interference experiments is conveniently understood by the use of Bratteli diagrams. These diagrams graphically describe the Hilbert space of ground states formed by the quasi-particles within the interference loop. The x–axis of the diagram enumerates the quasi-particles. The y–axis enumerates the *fusion channels* to which these quasi-particles may fuse. For the ground state subspace of n quasi-particles, each ground state corresponds to a trajectory that starts at the origin and goes n steps rightwards. The interference visibility and phase depend only on the final point of the trajectory, that is on the number of quasi-particles and their fusion channel. For the most experimentally relevant case of $\nu = 5/2$ the visibility vanishes when the number of quasi-particles in the loop is odd. When this number is even, there are two fusion channels. The visibility is the same for both of these channels (its actual value depends on various parameters such as temperature, the size of the interferometer, phase breaking processes etc.), but their phases differ by π.

Two types of interferometers have been proposed as "detectors" of non-abelian quasi-particles. A Fabry-Perot[23] interferometer is depicted in Fig. (3(a)) . It is a rather straightforward realization of the notions described above. The measured quantity is the current reflected back by the two constrictions. It is to be measured as a function of the magnetic field and the area of the "island" formed by the two constrictions. The number of quasi-particles trapped in the island is to be controlled by the magnetic field. It is assumed not to fluctuate in time. In the limit of weak tunnelling, the probability of a quasi-particle to be reflected back by the two constrictions takes the form

$$p = p_0 + p_1 \cos\phi \tag{2}$$

and the two-terminal conductance of the interferometer is $G = \left(2 + \frac{p}{2}\right)\frac{e^2}{h}$ (the two filled Landau levels are assumed to be perfectly transmitted). The amplitude p_1 of the oscillations of the conductance is determined by the parity of the number of

quasi-particles in the island, and is therefore turned on and off periodically as the magnetic field is varied. The phase ϕ is determined by the Aharonov-Bohm effect, by the abelian anyonic phase accumulated by quasi-particles as they encircle one another and by the quantum state to which the trapped quasi-particles fuse.

The Fabry-Perot interferometer with an even number of trapped quasi-particles may be used as a topological qubit, with the two possible fusion channels corresponding to the bit's "0" and "1". The measurement of the two-terminal conductance is to be used for a read-out of the qubit's state. This usage of the $\nu = 5/2$ state for topological quantum computation (TQC) does not employ the full power of TQC, since the set of possible unitary transformations that may be carried out by braiding $\nu = 5/2$ quasi-particles is limited.[21,24]

Many obstacles are likely to stand in the way for experimental realization of this experiment. Some are technical, such as the need for extremely good sample quality, very low temperature and highly controllable gates, but others are more conceptual. Some of the latter, such as the effect of thermal fluctuations in the number of island quasi-particles and the effect of coherent bulk-edge tunnel coupling, were discussed in the literature.[21]

The second type of interferometer is the Mach-Zehnder (M-Z) one,[25] depicted in (3(b)). Similar to the previous type, this interferometer partitions a current that comes from one source into two drains, with the partitioning ratio being determined by quantum interference. There are some differences, however, of which the most important one is that for the M-Z interferometer the interference loop includes the interior edge as well as one of the drains. As a consequence, the number of quasi-particles enclosed in the interference loop varies with each quasi-particle that tunnels from the exterior edge to the interior edge, and as electrons leave the edge through the ohmic contact. Since the probabilities p_{D1}, p_{D2} for quasi-particles to reach the drains D_1, D_2 depend on the number and the state of quasi-particles trapped in the interference loop, *these probabilities vary with each quasi-particle that tunnels between the exterior and the interior*. Thus, when a quasi-particle approaches a M-Z interferometer coming from the source, the probability that it ends up in a particular drain depends on how many quasi-particles have already tunnelled between the edges and on the state to which the quasi-particles on the interior fuse (electrons that leave the edge through the ohmic contact do not affect the interference, since

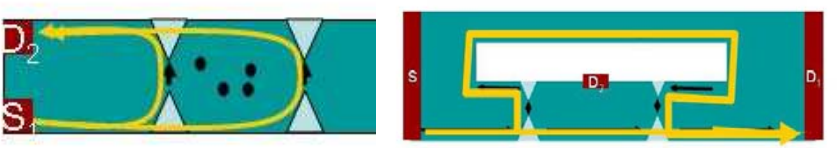

Fig. 3. (a) A Fabry-Perot interferometer in a Hall bar. (b) A Mach-Zehnder interferometer.

Eq. (2) is periodic with a period of four quasi-particles, i.e., a single electron).

In contrast to the F-P interferometer, where the partition of the current between the two drains is measured with the interference loop fixed to one point on the Bratteli diagram, for the M-Z interferometer the flow of current from the source on the exterior to the drain in the interior amounts to a motion on the Bratteli diagram. When the system is on one node of the diagram, it has a certain rate to hop to the adjacent nodes. This hopping rate depends on the bond along which the system hops and on the magnetic flux through the interferometer. Transport properties of the interferometer are calculated through kinetic equations that describe the motion along the diagram. Among these properties, a particularly striking signature of the non-abelian nature of the $\nu = 5/2$ state lies in the shot noise.

In principle, for a binomial process of current partitioning, in the limit where most of the incoming current is transmitted and a small part of it is back-reflected, the ratio of the current noise to the back-reflected current ("the Fano factor") allows for the extraction of the back-scattered charge. For a F-P interferometer, this ratio would give the charge of the quasi-particle that tunnels across the constriction, independent of the flux through the interferometer. This is not the case for the M-Z interferometer, as a direct consequence of the different rates of propagation along different bonds of the Bratteli diagram. In fact, it turns out that the Fano factor would in this case be flux dependent, with a period of one flux quantum through the interferometer, and would span the range between 1/4 to about 3 (in units of the electron charge). In comparison, the same experiment carried out on a $\nu = 1$ Integer Quantum Hall state should give a flux-independent Fano factor of 1 (since the interference pattern does not depend on the number of electrons on the interior loop), and when carried out on a $\nu = 1/3$ FQHE state it should give a flux-dependent Fano factor that ranges between 1/3 and 1.

These interferometric experiments probe the topological interaction of quasi-particles by having quasi-particles that flow along the edge encircle other quasi-particles, which are either in the bulk (for the Fabry-Perot case) or both in the bulk and on the interior edge (in the M-Z case). Another set of proposed experiments focuses on probing the properties of the edge. An example to that[26] is a proposal to study an edge of a non-abelian quantum Hall state coupled to a small quantum dot. The chemical potential of the dot is held close to its degeneracy point, where its energy with either N or $N+1$ electrons is degenerate. Its coupling to the edge allows electrons to tunnel back and forth. Signatures for non-abelian physics are found in the dot's capacitance and in the fluctuations of its charge. Remarkably, for the Read-Rezayi series of non-abelian quantum Hall states, which are candidate states for filling factors $\nu = 2 + k/(k+2)$, these signatures may be analyzed by means of a mapping of the system onto the problem of a two-level magnetic impurity (corresponding to the two charge states quantum dot) coupled symmetrically to a one dimensional electronic system of k-channels (the k-channel Kondo problem). At the point of charge degeneracy the dot's capacitance would show a logarithmic temperature dependence for the $\nu = 5/2$ state and a $T^{-(k-2)/(k+2)}$ for $k > 2$.

Another proposed experiment that probes the edge of a non-abelian quantum Hall state is one[27] in which an edge state that separates a $\nu = 2$ bulk from a $\nu = 5/2$ bulk is perturbed by two constrictions, which are far from one another. These constrictions allow electrons to tunnel from the vacuum, across the integer $\nu = 2$ state, into the edge state. This tunnelling is weak. A voltage is applied between the two constrictions and the Fano factor of the resulting current is measured. Despite the fact that the tunnelling particles are electrons, the Fano factor is expected to assume the value 1.5. The reason for this value is that when an electron tunnels through the first constriction into the edge, it excites both the charged edge mode and the neutral Majorana mode. These two excitations then move towards the second constriction, at the two different velocities that correspond to the two edge modes. When each of the two approaches the second constriction, it has a probability of inducing a tunnelling of an electron. Thus, there is a tendency for tunnelling events across the two constrictions to be correlated, and that results in an enlarged Fano factor.

All the examples we reviewed above propose ways to test whether quantum Hall states, most particularly the $\nu = 5/2$ state, are non-abelian. Looking through them, one notices the relative lack of bulk experiments. Bulk experiments were very instrumental in establishing the composite fermion picture of the abelian states within the lowest Landau level. We mentioned several experiments of this kind in the previous section. The search for signatures of non-abelian physics in bulk properties whose measurement is feasible is still ongoing. Very recently two such proposals, dealing with thermopower and with the dependence of the chemical potential and magnetization on temperature, were put forward.[28]

While the experimental demonstration of non-abelian states is a great challenge at present, it is not the only one. Among the others, I will note the need for a theoretical "toolbox" for performing calculations involving such states. Such a toolbox was developed by Read and Green[29] for the $\nu = 5/2$ state, through the description of the state as a super-conductor of composite fermions, i.e., as a Bose condensate of bosons made of two fermions each. Other non-abelian states may be viewed as condensates of bosons made of k anyons each. However, this viewpoint has so far not been translated to a calculational scheme.[21]

5. Quantum Hall Physics Gets Broader

Like many other fascinating physical phenomena, the quantum Hall effect originates in the world of semi-conductors. It has, however, crossed far beyond that realm of condensed matter physics. First, it has been observed in many two-dimensional systems, with the most recent ones being graphene[30] and polar oxide hetero-structures.[31] Second, it has been predicted, and is presently searched for, to occur in two dimensional systems of rotated cold bosonic atoms. If ever observed, the main interest in the atomic QHE will be in it being realized in a bosonic system. Numerical calculations conclude that most bosonic quantum Hall states will

be non-abelian.[32]

Among the many difficulties of observing a QHE of atoms, the absence of an atomic electric charge is probably the hardest one. Due to their charge neutrality, the Lorenz force needs to be replaced by a different force that is perpendicular and proportional to the atom's velocity. Rotation creates such a force in the form of Coriolis force. However, it is technically very hard to rotate at the needed frequencies. Other schemes for producing "Lorenz-type" forces were proposed on optical lattices, but have not been realized yet. Schemes which are simple to realize are in great demand, and are a theoretical challenge.

Going one step further from the the observed QHEs, there are two directions of research that have been highly influenced by the concepts of the field. The first is that of topological insulators, and the second is the search for microscopic models whose low energy physics is governed by a topological quantum field theory. I will make only brief comments on both.

Topological insulators[33] share a basic feature with the quantum Hall effect: they are gapped in the bulk, but gapless at the edge. Just as for the quantum Hall effect, the gapless mode at the edge is a consequence of the bulk having a topological quantum number that is different from that of the vacuum. The transition from the bulk to the vacuum value must then involve a closing of the gap. Topological insulators differ from the QHE, on the other hand, by being symmetric to the reversal of the direction of time. As such, they cannot have chiral gapless modes. Whenever they have gapless modes, then, these modes have branches moving in opposite directions. Their stability to scattering of electrons between these branches may be analyzed by the same methods that were developed to analyze the stability of counter-moving edge states in the QHE.[34] For non-interacting electrons in two dimensions, the edge states may always be gapped when subjected to inter-branch scattering that breaks time reversal symmetry, but are sometimes stable to inter-branch scattering that does not break that symmetry, explicitly or spontaneously.

Remarkably, the topological insulators carry the physics of the quantum Hall effect over to the third dimension. While a three dimensional quantized Hall effect has so far been only a theoretical construct, a three dimensional topological insulator seems to have been been experimentally observed. Its response properties, the properties of its surfaces, and the fate of these surfaces when time reversal and charge conservation symmetries are absent are under intense present study. So are the mathematical classification of these systems.

The search for other microscopic models whose low energy physics is "quantum Hall like", and in particular "non-abelian quantum Hall like", turns out to be a very challenging task.[21] While several lattice models with anyonic excitations have been proposed,[21] none is close to experimental realization. Furthermore, at present we are far from understanding what are the basic requirements that the interactions in a microscopic Hamiltonian should satisfy for the low energy physics to be described by a topological quantum field theory.

6. Summary

For a summary, I invite the reader to join in for an admiring look at the beauty of the quantum Hall effect. There are many vista points from which this beauty may be admired. I choose one of a theorist: there is little doubt that the QHE is well described by the following electronic Hamiltonian:

$$H = \int d\mathbf{r} \ \left\{ \tfrac{12}{2m} |(i\nabla - \tfrac{e}{2c}\mathbf{B}\times\mathbf{r})\psi(\mathbf{r})|^2 + V(\mathbf{r})n(\mathbf{r}) \right\} + \tfrac{1}{2}\int d\mathbf{r} \int d\mathbf{r}' \, (n(\mathbf{r}) - n_0) \, \tfrac{e^2}{|\mathbf{r}-\mathbf{r}'|} \, (n(\mathbf{r}') - n_0) \tag{3}$$

with $n(\mathbf{r}) \equiv \psi^\dagger(\mathbf{r})\psi(\mathbf{r})$ the electron density, and $n_0, V(\mathbf{r})$ the positive background density and the (weak) disorder potential, respectively. This Hamiltonian has a tunable dimensionless parameter, the filling factor $\nu \propto n_0/B$. Along the axis where this parameter is tuned from zero up, the ground state of the system explores the Wigner crystal, the abelian and non-abelian FQHE, the IQHE, and the states of stripes and bubbles. The precise position of these phases along this axis is determined by the details, especially by the properties of $V(\mathbf{r})$. But the immensely rich web of unique properties shown by some of these states – the fractional charge and statistics of the excitations, the degeneracy of the ground state, the resistivity matrix, to name but a few – is blind to details, and controlled by topology. Within physics, it is hard to imagine a more intricate combination of simplicity and complexity.

Acknowledgments

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Discussion

J. Chalker I suggest we take a few minutes discussing points directly arising from Ady's talks.

S. Das Sarma I wanted to make two related points. One is that you mentioned that, first thing we need to make sure is that we have non abelian states. Let us stick to the Moore-Read state that describes 5/2 to start with. And you said that bulk experiments are hard to think of given that there is more than one candidate state in the bulk. But I think that there is one experiment that establishes the spin polarization properties of the system. So if we know that the experimental 5/2 state is fully spin polarized, and we already know that the charge quasi particle is e/4 from some beautiful experiments in your institution, then I would say that the Moore-Read nature, or at least close to something like that, is more or less well established because it is very hard to think of any other state that is fully spin polarized. Because candidate alternative states, like you know what they are, all would not have full spin polarization. So I personally think that measuring the spin polarization of 5/2 is a very important bulk experiment which is much easier than these edge experiments. And the related point is that we should also look for Moore-Read states at other fractions, for instance $\nu = 1/2$. You did mention that $\nu = 7/2$ is also is a fractional quantum Hall state, obviously described by the Moore-Read state if 5/2 is. But $\nu = 1/2$ would be a candidate non abelian state if we manage to find a system with a fractional quantum Hall incompressible configuration.

A. Stern Of course this is a matter of taste. If indeed the state is found to be fully spin polarized then there will be what is called circumstantial evidence, and you do not send somebody to jail for that, circumstantial evidence, but that will more or less say indeed that we do not have better candidates than the Moore-Read state but I think that I would still like to see an experiment directly showing that the state has the some property that is predicted by the Moore-Read theory.

S. Das Sarma But what such evidence do we have that 1/3 is such a Laughlin state?

A. Stern OK. Yes of course. In my talk there was a piece on fractional statistics which I decided not to include because of lack of time. We are still in the need of proving the nature of the 1/3.

B. Altshuler I just wanted to ask the experts a question that always bothered me in connection with these non abelian states. What is a localized quasi-particle. I mean when you apply potential you apply to electrons, not to quasi-particles, which looks like what you will localise is a linear combination of one quasi-particle, 3, 5 and so on. Now when you will manipulate with that you can carry all these states in slightly different paths. Will it cause problems? Is it a problem but a resolvable one? What is the status?

A. Stern So that may be a place where I can refer to experiments. We know that in abelian states, in the 1/3 state, scanning experiments of my past colleague Amir Yacoby showed that charging is done by units of 1/3 of a charge. So if your electrostatic trap is such that the charging energy process is just 1/3 of a charge, the Laughlin liquid allows to do that. And that is a partial proof that Laughlin is good. So if Amir does the same type of experiment and sees 1/4 then we are fine. And I see no reason why it should not happen.

X.-G. Wen I have two comments. One is a response to Shankar's comment. Theoretically I think it is possible to construct, you know, 4 or 5 different spin polarized filling fraction 1/2 non abelian quantum Hall states. But many of them may not be very likely theoretically, but still existing, you know such things exist. They all have a charge 1/4, so it is crucial to measure some kind of external exponent from which you can get more information than charge, some information about statistics. And the second comment is that you mentioned a lot about the edge states. I think the crucial difficulty in edge experiments is that the quantum Hall edge is pretty smooth. You know there is a region of a few thousand Angstrom where the density gradually goes from quantized value to zero, which is much bigger than magnetic length. So I think all this theory is really about sharp edge which is a density drop over the order of the magnetic length. So there is a kind of gap here. To overcome this gap is a huge challenge.

A. Stern Yes. I agree. But we cannot fully throw this problem to the area of crystal growth and fabrication. Because the experiments I was showing of tunnelling are experiments of cleaved edge. That is a sharp as you will get.

X.-G. Wen No. The cleaved energy is the energy band of the full upper band by an order eV. Only the tail will give you meV which is about Fermi energy of 2D gas. So even for a chemical cleaved edge, you have to assume that ... by milli electron volts level, then you have a sharp edge otherwise it is still a huge energy.

A. Stern No the point I want to make here: the challenge is ours.

M. Freedman Also a comment on the value of direct measurements of the non abelian charges. Since there is a possibility with non abelian particles to measure

their statistics rather more directly than abelian phases, it seems this should be the focus of experiment rather than indirect measurements because, ultimately, we do not just want to measure, we want to be satisfied that it is non abelian and go home. We want these measurements not only to be possible but easy. We want to be able to do them a thousand times a second, you know to run a computer. So if we have indirect evidence that will not help us manipulate the state in a useful way for computation.

A. Stern I fully agree.

B. Halperin I would just like to mention a couple of closely related points. The first is in response to Boris's question. If you have a region that may be localized in some trap, where you have perhaps some quasi particles, and it is surrounded by a region that is reasonably wide, where it is sufficiently exactly close to the ideal filling fraction that there are no quasi-particles, then you have a tunnelling barrier from inside to outside, and you get quantization of the charge inside. This is what Amir found. You can add 1/3 charged quasi particle and see the charge jump by 1/3 and so forth. Now for the non abelian statistics it is a bit more delicate because not only do you have the possibility of the quasi-particles tunnelling across, which would change the number of quasi particles inside this ring of pure system, you also have the possibility of Majorana fermions tunnelling across, which might happen more easily. And if the Majorana fermions can tunnel across in some reasonable time then that means you can change the internal state, which will change the phase of the interference signal by π. If this happens many times during the course of a single measurement, then time-averaging will destroy the interference signal. So getting nice clean regions of some size that are good enough that you can ignore the exponentially small tunnelling between localized regions is essential for any of this study of the non abelian statistics. And that's quite a challenge.

B. Altshuler But this size should not be exponentially big, should it?

B. Halperin No, it need only be logarithmically big. That is, once you get to the point where the exponenential factor in the tunneling rate is, say exp (-20), you only have to increase the separation between quasiparticles by another factor of two or four before you get really small tunneling rates. However, you first have to get to exp (-20). Even exp (-2) may not be trivial to obtain. We do not know what is the correct decay length for the exponential, but probably it is a couple of times the magnetic length, at least, which would be around 20 nm. So that means your quasi-particles may have to be very far apart, if you want to explore the non-abelian statistics.

A. Stern There are several goals here. If you want a qubit like Mike was talking about, then definitely you need to have the bulk and the edge very well separated from one another. If you want just to show non abelian statistics, and you look just at the Mach-Zehnder problem, since each of these Majorana fermions tunnelling or charge fluctuating will make you diffuse on that diagram that I was showing, in a Mach-Zehnder interferometer you any way diffuse, you any

way propagate. So this type of charge noise will not spoil the experiment, it will still give you the shot noise I was quoting. The signature will not disappear. That is the advantage of this type of interferometer.

J. Chalker OK. I think that at that stage it might be good to move on to a prepared comment about experiment by Jim Eisenstein.

Prepared Comment by Jim Eisenstein: Prospects for the Quantum Hall Effect

1. Introduction

Broadly defined, the quantum Hall effect (QHE) deals with the phenomena exhibited by clean two-dimensional electron systems at high magnetic fields. This field has, and remains, one of the most vibrant in modern condensed matter physics. Its past successes have fundamentally altered the way we think about the quantum many-body problem. The scope of the field continues to expand, with important connections to fields as disparate as ultra-cold dilute atomic gases and classical liquid crystals developing in recent years. New systems exhibiting quantum Hall phenomena continue to appear; atomic monolayers of carbon are a notable recent example. In this brief contribution I will offer a personal perspective on some of the outstanding opportunities and challenges facing the field today.

2. Opportunities

2.1. *Anyons: Abelian and otherwise*

Fractional quantum Hall states are believed to harbor an unusual class of quasiparticles called anyons. Unlike the ordinary elementary particles, anyons are neither fermions nor bosons. Their charge is a fraction of the elementary electron charge and interchanging two of them multiplies the many-particle wavefunction not by plus or minus one, but by a complex phase. While there is strong evidence that the charge of anyons is indeed fractional,[1] the direct observation of their peculiar exchange statistics remains an elusive, yet extremely appealing goal for the field.

A bizarre subclass of anyons is expected to exhibit non-abelian exchange statistics.[2] Multiple pairwise interchanges within an ensemble of non-abelian anyons does much more than multiply the system wavefunction by a phase; it creates a *distinct quantum state* dependent upon the order in which the interchanges took place. The ground state of a system of many non-abelian anyons in fixed positions is therefore highly degenerate. This strange property, not possessed by fermions, bosons, or even ordinary abelian anyons, suggests very tempting opportunities for quantum information science. At present the most likely place to search for non-abelian anyons is in the so-called 5/2 fractional quantum Hall state. Ady Stern's contribution to

these Proceedings describes the 5/2 state and its possible application to quantum computation in detail.

2.2. *Edges*

In a quantum Hall state the bulk of the 2D electron system is an insulator. Ideally, charge transport through the system can only occur at the boundaries of the system. At these boundaries fascinating one-dimensional electronic states exist. In the integer QHE these edge states are easily traced to the various single particle Landau energy levels created by the magnetic field. In the fractional QHE the situation is much more subtle. Here electron-electron interactions are all-important and there is no simple backdrop from which edge states naturally emerge. Nonetheless, there are by now sophisticated theories of the fractional quantum Hall edge. Unfortunately, many of the predictions of these chiral Luttinger liquid (CLL) theories[3] have so far gone unverified by experiments. Prominent among these is the prediction of charge neutral edge modes which propagate along the edge in the direction opposite to that followed by ordinary electrons.[4,5]

The CLL theory is best understood for ideal, sharp edges where the electron density rapidly drops to zero. This is not a good approximation to most experimental situations in which the electron density subsides over hundreds of nanometers. Such smooth edges may be "reconstructed" and thereby support several different kinds of edge modes. Arriving at a better understanding of the fractional quantum Hall edge is obviously important for testing the validity of CLL theories. It is also likely to be important for developing mesocopic devices (e.g. interferometers) designed to test and exploit the properties of non-abelian quasiparticles for quantum computation.

2.3. *Graphene*

It has recently been found possible to isolate and study single atomic layers of graphite.[6] This new material, called graphene, has an unusual gapless band structure consisting of linearly dispersing conduction and valence bands, in close analogy to a system of massless relativistic particles. Aside from its many remarkable electronic, thermal, and mechanical properties, graphene exhibits the quantum Hall effect. To date, only integer quantized Hall plateaus have been observed. While these are certainly interesting, and may involve electron-electron interaction effects at some level, fractional quantum Hall states would be even more interesting. The absence of such states, in all reported samples so far, including those in which the graphene layer is suspended in vacuum between supports, is presumably due to the still large amounts of disorder in these ultra-thin films.

It is clearly desirable to develop new methods for producing graphene. The present mechanical exfoliation method is limited to the production of isolated graphene flakes at random positions on a substrate. Engineering more complex structures, analogous to the multilayer semiconductor heterostructures, is not yet possible. Even the simplest such system, two graphene layers separated by a con-

trollable barrier, would be fascinating to study. Interlayer tunneling and drag effects could be examined and exotic new states of matter (e.g. an excitonic superfluid at zero magnetic field) might be found.

3. Challenges

3.1. *Crystal growth*

Progress in the quantum Hall effect field is intimately tied to progress in crystal growth, specifically molecular beam epitaxy (MBE). Following the invention of modulation doping,[7] the mobility of two-dimensional electron systems (2DESs) in GaAs/AlGaAs heterostructures began a remarkable, decades long rise. The fractional quantized Hall effect was first observed in 1982 in a sample with low temperature mobility $\mu \approx 10^5$ cm^2/Vs. By 2000 the mobility of similar structures had risen to roughly 3×10^7 cm^2/Vs. This rise was not steady, being punctuated instead by a series of steps upward followed by pauses. Each of these steps resulted in one or more important discoveries. For example, in the mid-1980s the mobility had reached a few million cm^2/Vs. At this level the first convincing evidence for a fractional QHE at Landau level filling factor $\nu = 5/2$ was reported.[9] By the early 1990s the mobility was around 10^7 cm^2/Vs. This level of purity allowed for dramatic verifications of the composite fermion model for the half-filled lowest Landau level.[10] Late in the same decade, with the mobility approaching 2×10^7 cm^2/Vs, an entirely new class of collective 2D electron phases, the so-called stripe and bubble phases, was discovered.[11,12] These experimental discoveries, along with several others not mentioned, were paralleled by major advances in theoretical condensed matter physics.

While mobility is the simplest measure of the advance of MBE crystal growth, it is important to recognize a different measure of progress: structural complexity. One example of this is the so-called cleaved edge overgrowth method.[13] This very difficult growth technique allows for the realization of nearly perfect one-dimensional (1D) electron gases within sharp confinement potentials. Among the several fruits of this technique has been the observation of spin-charge separation in 1D systems.[14] Another example is the growth of bilayer 2D electron systems in double quantum wells. While such structures are easy to envision, it has proven remarkably difficult to create them with high electron mobility. In spite of this, bilayer quantum Hall systems have revealed some extremely interesting phenomena, most dramatically a novel coherent state which may be viewed as a Bose condensate of excitons.[15]

3.2. *Low temperatures*

As the mobility of the 2D electron system is increased and the random disorder potential experienced by the electrons is reduced, ever more subtle and interesting collective effects become detectable. The progress of the field depends upon this relation. But "subtlety" here translates into low temperature and we have reached a significant barrier in that regard. For the last twenty years or so it has been

sufficient to employ commercial dilution refrigerators for research in the field. These systems can routinely cool bulk matter to temperatures as low as about 5 milli-Kelvin. In quantum Hall systems however, it is not sufficient to simply cool the host semiconductor crystal; it is the electron temperature which counts. Here one finds things difficult since the very dilute electron system only weakly communicates with the phonons of the host material. Indeed, below about 100 mK cooling of the electrons relies more on electronic diffusion to the ohmic contacts embedded in the host crystal than on direct cooling via electron-phonon coupling. In this way experimentalists have been able to reach electron temperatures of around 15 mK.

If the fruits of mobilities in excess of about 10 million cm^2/Vs are to be fully realized, we must cool electrons to well below 10 mK. Already there is evidence that such low temperatures are essential. Some recently discovered phenomena only barely show up at 15 mK; one can only guess what lies even closer to absolute zero.

Cooling to much below about 10 mK (with sufficient cooling power) requires a different technology than dilution refrigeration. Adiabatic nuclear demagnetization and Pomeranchuk refrigeration are two methods for reaching down to 1 mK and below, in bulk matter. These methods, while long established, are not readily available off-the-shelf, and setting them up is a major undertaking. More importantly, the problem of communicating the very low temperatures they create to the 2D electron gas inside the semiconductor host is an enormous challenge. Indeed, this is the real nub of the problem. Some progress has been made and there have been a handful of demonstrations of electron temperatures in the few mK range.[16] But much more work is needed before these methods can be widely and reliably applied. Without it, possessing a 2D electron gas with a mobility of 100 million cm^2/Vs (an avowed goal of crystal growers) will be of limited value.

4. Conclusion

The quantum Hall effect field has remained robust for nearly three decades. Its impact on condensed matter physics is hard to over-estimate. From its dramatic relevance to metrology to the recent intense examination of its potential for enabling quantum computation, the field has maintained a technological dimension. The path to understanding quantum Hall phenomena has also created numerous new concepts in theoretical physics, including charge and statistical fractionalization, flux attachment and Chern-Simons composite particles, non-abelian exchange statistics, and so on. Nowadays quantum Hall phenomena and concepts are becoming relevant to fields of physics seemingly far removed from humble electrons confined to a plane surface. Great opportunities and formidable challenges beckon.

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Discussion

J. Chalker Questions and Comments?

T. Senthil What are the prospects for measuring the thermal Hall conductivity?

J. Eisenstein I do not know. I have not thought about that.

X.-G. Wen I think that from experimental side a double layer system may offer another route to non abelian quantum Hall states. You know in double layer systems you can have excitons and what is very special about excitons in quantum Hall effect is that exciton may have fractional statistics. So the condensation of excitons will be like a condensation of neutral anyons which offers very interesting theoretical and experimental possibilities. That is certainly an interesting direction.

M. Fisher Just a comment. I think a conceptual frontier on the theoretical side where the real things which have not been started to explore are in multi layer quantum Hall systems like three dimensional quantum Hall systems like in bulk bismuth or in graphite where one has significant tunnelling of the electrons between the layers so they are really more three dimensional as far as their electrical transport and it seems pretty clear that there are whole new sets of three dimensional fractional quantum Hall effect type phases which have not been explored theoretically. And on the theoretical side what is interesting to me is that all the techniques which have been used, Chern-Simons theory in particular, flux attachment, really go out the window as soon as you have a bilayer where the electrons can tunnel between the layers. In three dimensions we can not statistically transmute electrons and bosons in the same way we do in two dimensions. And I think there is significant scope for interesting experiments on bi-layers where the electrons are tunnelling significantly between the layers, order one tunnelling between the layers. There are new phenomena to be discovered experimentally hopefully and to explore theoretically, and there

are issues we are just starting to grapple with.

J. Eisenstein I add one comment to your comment, and that is that double layers are really two single layers put together back to back, and you could do that easily by putting the dopants far away from the two dimensional system. As soon as you go to many layers you are in trouble to give carriers into the interior quantum wells without having very very low electron mobility.

M. Fisher I understand that. So I think in the two dimensional electron gas context two layers, perhaps three can be where you want to stop. But already in two layers there will be interesting phenomena. But already in graphite or graphene, in multilayers, bismuth and potentially other systems which are more three dimensional where one will not have that same issue.

M. Cohen Another way to achieve the indirect exciton and exciton condensation is to look at two nanotubes that are in close proximity. In a perpendicular electric field you can control the gap and you can show theoretically that you can have exciton condensation. These are indirect excitons in the same sense you are talking about. Experimentally the way to do this is to work with nanotube bundles which occur naturally and put on a perpendicular electric field after creating the excitons.

J. Chalker I thought we should have a prepared contribution more on the theoretical side and I believe Nick Read has something ready.

Prepared Comment by Nick Read: Quantum Hall Fluctuations, Conformal Blocks and Topological Phases of Matter

1. Abstract

Many trial wavefunctions for fractional quantum Hall states in a single Landau level are given by functions called conformal blocks, taken from some conformal field theory. Recently, the adiabatic transport of such many-particle trial wavefunctions has been studied using methods from two-dimensional field theory, to calculate the statistics of widely-separated quasiholes, which may be non-Abelian. Necessary and sufficient conditions for the adiabatic statistics to be the same as analytic continuation of the wavefunction have been found, and these conditions are likely to hold in many cases. We argue that trial states based on a non-unitary conformal field theory do not describe a gapped topological phase. We also discuss the *Hall viscosity*, a non-dissipative viscosity coefficient analogous to Hall conductivity.

2. Introduction

In this contribution, I will review some aspects of quantum Hall (QH) phases of matter not covered elsewhere in these proceedings, including the use of conformal blocks from conformal field theory (CFT) as trial wavefunctions.[1] This point of view

provided the original motivation and examples for the proposal that the statistics of excitations in quantum Hall systems could in some cases be non-Abelian. Topological (i.e. gapped) phases of matter with non-Abelian excitations are the tools required for physical implementation of topological quantum computation.[2] It is widely believed, though it has not been directly demonstrated, that the state observed at filling factor 5/2 is in the Moore-Read non-Abelian phase of Ref. 1, or its particle-hole conjugate. While aspects of some non-Abelian QH phases can be understood by other means (e.g. using techniques for paired superfluid states of fermions[3]), we will see that two-dimensional quantum/statistical field theory ideas are still useful, for example for precise calculations[4] of adiabatic transport of trial states (as required for statistics of excitations), which remain valid throughout a topological phase of matter. The resulting characterization of topological phases through the mathematics of modular tensor categories will lead us to striking conclusions about the use of conformal blocks from non-unitary CFTs as trial wavefunctions: the associated Hamiltonians are most likely gapless (or critical).[4]

3. Trial Wavefunctions and QH Phases

Let us begin with Laughlin's wavefunction for a ground state plus quasiholes,[5] in a version (essentially) proposed by Halperin:[6]

$$\Psi(w_1,\ldots,w_n;z_1,\ldots,z_N) = \prod_{k<l}(w_k-w_l)^{1/Q}\cdot\prod_{i,k}(z_i-w_k)\cdot\prod_{i<j}(z_i-z_j)^{Q}\cdot e^{-\frac{1}{4Q}\sum_k|w_k|^2-\frac{1}{4}\sum_i|z_i|^2} \quad (1)$$

Here z_i ($i = 1, \ldots N$) are the complex coordinates of the particles, and w_k ($k = 1, \ldots, n$) are the complex coordinates of quasihole excitations (these play the role of parameters in the wavefunction for the particles). Q is an integer, and for Q odd, the particles are fermions (perhaps electrons), and for Q even bosons (perhaps atoms in a trap). The particles are in the lowest Landau level (LLL), and the magnetic length has been set to 1. The particles cover a roughly circular region in the plane, and the filling factor, a dimensionless measure of the particle density inside this region, defined in the absence of quasiholes ($n = 0$), is $1/Q$. A net charge of $1/Q$ (in units of particle number) is missing from the vicinity of each w_k (provided they lie inside the region), which is the celebrated fractional charge of the quasiparticles. The factors that are independent of the z_i's could clearly be omitted, but in the form shown the functions are normalized (up to a constant) independent of the w_k's, provided their separation is sufficiently large (thus requiring a sufficiently large number of particles also). All of the preceding statements can be established by using Laughlin's mapping of the modulus square wavefunction to a Coulomb plasma with uniform background charge density in two dimensions.[5] The factor $\prod_{k<l}(w_k - w_l)^{1/Q}$ includes a phase (not needed for normalization) such that it is holomorphic away from the diagonals on which $w_k = w_l$ for some k, l; this factor will play a role in the following discussion. Finally, we note that these states are zero-energy eigenstates of a local two-body interaction Hamiltonian that acts within

the LLL,[7] and there seems to be a gap for all other excitations above these states in the thermodynamic limit, if Q is not too large.

Next we introduce the ground state wavefunction of Moore and Read (MR):[1]

$$\Psi_{\mathrm{MR}}(z_1,\ldots,z_N) = \mathcal{A}\left(\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots\frac{1}{z_{N-1}-z_N}\right)\prod_{i<j}(z_i-z_j)^Q\cdot e^{-\frac{1}{4}\Sigma_i|z_i|^2}, \quad (2)$$

where the antisymmetrizer $\mathcal{A}$ sums over all permutations that produce distinct pairings (i,j), times the sign of the permutation. Here $Q>0$ is odd for bosons, even for fermions, and the filling factor is again $1/Q$. It was pointed out by MR that this wavefunction can be viewed as the real-space form of a BCS[8] $p+ip$ paired state of composite fermions (which are in zero net effective magnetic field at these filling factors). The paired order parameter immediately suggests that the quasiparticle excitations, analogous to those of Laughlin, are associated with a *half* flux quantum, and so will carry charge in multiples of $1/2Q$. The excitations with charge in odd multiples of $1/2Q$ were predicted to obey non-Abelian statistics, for reasons we will discuss later. In addition, there will be neutral fermion excitations (BCS quasiparticles). In a more detailed study of the pairing point of view,[3] it was shown that a $p+ip$ paired state of spinless or spin polarized fermions can be in a phase in which there is a Majorana (i.e. self-adjoint) fermion operator associated with each half-flux vortex, and these operators create or destroy states that have zero excitation energy when the vortices are well separated. The resulting degeneracy of states, namely $2^{n/2-1}$ for n vortices when the particle number is fixed, provide the subspace in which non-Abelian statistics can act. Finally, there is again a LLL Hamiltonian for which the trial wavefunctions are zero-energy eigenstates, as outlined in Ref. 9. In this case it contains three-body local interactions. The corresponding QH trial state wavefunctions were constructed explicitly,[1,10,11] and the quasihole states exhibit the expected degeneracies.

4. Adiabatic Transport

The physically-meaningful way to obtain the statistics of excitations is by adiabatically transporting them until they are returned to the original positions up to a permutation of identical quasiparticles. In the most general setting, there may be a space of degenerate states which depend on a parameter w (or set of parameters w_l), and then the adiabatic transport (in the space of w, starting and ending at a base point w_0) produces not a Berry phase, but a unitary matrix.[12] If ψ_a is an orthonormal basis set for this space (for us, the norm-square of a wavefunction Ψ is the usual one $\langle\Psi|\Psi\rangle = \int\prod_{i=1}^N d^2z_i\,|\Psi|^2$), then the effect of adiabatic transport is to map $|\Psi_b(w_{(0)})\rangle$ to $\sum_a|\Psi_a(w_{(0)})\rangle B_{ab}$, where the Berry matrix ("holonomy") is given by

$$B = M\mathcal{P}\exp i\oint_C(A_w\cdot dw + A_{\overline{w}}\cdot d\overline{w}). \quad (3)$$

Here the Berry connection is given in components by

$$A_{w,l,ab}(w) = i\left\langle \Psi_a(w) \left| \frac{\partial \Psi_b(w)}{\partial w_l} \right. \right\rangle, \tag{4}$$

and the unitary matrix M is often required to transform the basis states at the end back to the original basis and gauge. In this form, the holonomy is gauge-covariant under unitary changes of basis set or change in the base point w_0.

The following observation is useful: if the states (wavefunctions) Ψ_a are holomorphic in w as well as orthonormal, then

$$A_{w,l,ab}(w) = i\frac{\partial}{\partial w_l}\left\langle \Psi_a(w) \middle| \Psi_b(w) \right\rangle = 0, \tag{5}$$

and so $B = M$. Thus it is useful to have trial wavefunctions that are holomorphic in the parameters w_l, as is the case for the Halperin form (1) above (the neutralizing background factors, which are Gaussians, are not holomorphic in w, but can be easily dealt with; we neglect such aspects below). As these states are also orthonormal (up to w-independent constants), and the matrix M reduces to a phase factor which is $e^{i\pi/Q}$ for an exchange of two quasiholes along a path that does not enclose any other quasiholes, it follows that the (adiabatic) fractional statistics of the quasiholes for such an exchange is also $e^{i\pi/Q}$, reproducing the result in Ref. 13.

5. Trial Wavefunctions as Conformal Blocks

In order to calculate the adiabatic statistics for excitations over trial states such as that of MR, we introduce the point of view of MR concerning conformal blocks viewed as trial wavefunctions.

Conformal blocks arise in two-dimensional (2D) CFT,[14,15] which describes both massless quantum field theories in $1+1$ dimensions and phase transitions in two space dimensions. In these theories, correlation functions ("correlators") factorize into holomorphic and antiholomorphic parts, for example, for a correlator of two fields ψ and τ, this would take the form

$$\langle \psi(z_1,\bar{z}_1)\cdots\psi(z_N,\bar{z}_N)\tau(w_1,\bar{w}_1)\cdots\tau(w_1,\bar{w}_1)\rangle_{\mathrm{CFT}} = \sum_a \left|\mathcal{F}_a(w_1,\ldots;z_1,\ldots)\right|^2, \tag{6}$$

where $\mathcal{F}_a$ are (a basis for) the blocks, which are labeled by an index a (the range of which depends on both the theory and the correlator). The blocks span a vector space, and the basis we have chosen is one in which the right hand side is this simple sum of squares. The conformal blocks are holomorphic functions of their arguments locally, but generally have branch cuts on the diagonals, at which two of the z_i's or w_l's coincide.

The Halperin forms (1) are conformal blocks in a free scalar field theory (the Gaussian factors are produced by a uniform background[1]). The MR ground state arises from a similar Laughlin factor times a conformal block from the CFT of the 2D Ising model. That is, the paired factor in (2) arises from a Majorana fermion operator at each position z_i. (The blocks themselves can be obtained from factoring the

operators ψ or τ as e.g. $\psi(z,\bar{z}) = \psi(z)\overline{\psi}(\bar{z})$, and it is the chiral operator $\psi(z)$ that is used here.) For the trial wavefunctions, the fields ψ at the particle coordinates are chosen so that when the block is combined with s suitable Laughlin-Halperin factor, the wavefunction is single-valued and holomorphic everywhere as a function of the z_i's, with the w_l's fixed. The coordinates w_l are the positions of quasiholes, and the wavefunctions are not usually single-valued in these variables. Instead, like the conformal blocks, they exhibit monodromy: as the functions are analytically continued in the w_l's (with z_i's held fixed), they are not single-valued but on traversing a circuit with base point w_0 they return to themselves up to a matrix M (in the basis of the $\mathcal{F}_a$) which is independent of w_0 up to a unitary transformation. A series of further examples is given in Ref. 16.

A basic idea of MR is that the Berry connection arising in adiabatic transport of these wavefunctions vanishes, so that the adiabatic statistics (holonomy) B is given by the monodromy M, just as in the case of the Laughlin-Halperin quasihole functions. We will argue that a necessary and sufficient condition for this is that the system be in a topological (gapped) phase (in the $2+1$ sense), or that a certain 2D field theory be in a massive phase.

6. Perturbed CFTs

Because of the remarks in Sec. 4, and because the trial wavefunctions are conformal blocks which are holomorphic for well-separated quasiholes, we only need to examine the overlaps $\mathcal{Z}_{ab} = \langle \Psi_a | \Psi_b \rangle$ of the trial wavefunctions Ψ_a. Because the conformal blocks are correlators of chiral fields like $\psi(z)$ and $\tau(z)$, these overlaps resemble the *non-chiral* correlators (6). They differ in three ways (i) the wavefunctions also include the Laughlin-Halperin factors for the charge sector; (ii) the correlators are summed over $a = b$; and (iii) the overlaps are integrated over the coordinates z_i. For the charge sector factor, we know that this is also a conformal block, so this does not change the situation (another field is multiplied into ψ and τ). For point (ii), let us first consider only the sum of overlaps (trace of $\mathcal{Z}_{ab}$). Finally, for point (iii), we point out that such integrals of correlators like (6) occur when a CFT (or critical point) is perturbed by a perturbation like $S_1 = \int d^2\psi(z,\bar{z})$. This is usually understood as adding the perturbation to the action, so the perturbed correlator of τ's is $\langle e^{S_1} \cdots \rangle$. The correlator we have is just one of those that result from expanding the exponential. As in statistical mechanics, the difference between these "canonical" and "grand-canonical" points of view should not be important when we calculate these relatively local correlation functions.

We therefore consider the flow of the underlying CFT under the perturbation, all as 2D field theories.[4] The main point is to determine the ultimate renormalization group (RG) fixed-point reached in the infrared (at large distances, relevant in particular for large separations of the quasiholes). It will be important to determine whether it is massive (as a 2D theory), or massless. For the Laughlin states, the long-distance fixed point is massive (if Q is not too large, $Q < 70$), meaning that

the plasma is in a screening phase. Most physicists will agree that the same is true for the charge sector for the other trial wavefunctions also. Then we only have to consider the perturbation of the remaining CFT (the "statistics" sector). If this flows to a massive phase, then all correlations (such as those of τ) approach their long-distance limits with exponentially decaying corrections. This generalizes the screening property of the Laughlin states (2D plasma) to the general case. If the limit is non-zero, then it should be constant, as it can be interpreted as the factorization of this limit of the correlator into a product of expectations (by clustering). (This part of the argument is similar to one in Ref. 10.) Generically, the limit of a correlator of the scaling fields of a critical point under a perturbation that flows to a massive phase does go to a non-zero value, except when disallowed for reasons we discuss next.

This almost completes the argument, but we must still remove the summation over $a = b$ in the overlap. We ask whether all the matrix elements of $\mathcal{Z}_{ab}$ are allowed to be non-zero. Now the monodromy of the trial functions (blocks) implies that $\mathcal{Z}_{ab}$ transforms as

$$\mathcal{Z}_{ab} \to \sum_{c,d} (M^\dagger)_{ac} \mathcal{Z}_{cd} M_{db}, \tag{7}$$

for example under an exchange of quasiholes by analytic continuation. But this is not consistent with the constancy of the overlap unless $\mathcal{Z}_{ab} = \sum_{c,d} (M^\dagger)_{ac} \mathcal{Z}_{cd} M_{db}$ under all possible exchanges. If, say, the monodromy yields an irreducible representation of the braid group, then using Schur's lemma this implies that the overlap matrix $\mathcal{Z}_{ab}$ is proportional to the identity, and the trial states are orthonormal (up to a constant), and so the holonomy equals the monodromy. That is, the adiabatic statistics can be read off from the conformal blocks. If the representation is not irreducible, then the argument still gives sufficient constraints on the overlap to yield the same result.

If the 2D field theory does not flow to a massive phase, but to a massless (critical) one (the massless phase may also be conformal), then there are several difficulties. On the face of it, a massless case can produce a non-zero Berry connection that makes the adiabatic statistics different from the monodromy of the blocks. However, one of the difficulties with this is that such RG flows generally approach their limits with power-law corrections due to the decay of irrelevant operators. In the present case this translates into subleading power corrections to the Berry connection, which do not correspond to the behavior of a topological phase (in $2 + 1$ dimensions). Instead they suggest gapless behavior.[4]

These two possibilities (massive and massless 2D behavior) appear to be the only possibilities (leaving aside worse behavior such as overlaps tending exponentially to zero or infinity, which introduce a length scale into the Berry connection, again not the behavior of a topological phase). It is striking that the only adiabatic statistics that can arise in a topological phase represented by using conformal blocks as trial wavefunctions is the same as the monodromy of the blocks.

7. Topological Phases of Matter

It is of interest to take the present approach further to fully characterize the resulting 2 + 1 topological phases. There is a mathematical framework that captures the essential properties that characterize a topological phase of matter, the theory of modular tensor categories (MTCs).[17] (Actually, this framework is appropriate when the particles are bosons, but needs to be extended to include the situation in which the particles are fermions, such as electrons. Work on this is in progress, but from a physical point of view the two cases are always similar, and there should not be a major problem.) The structure needed to define a MTC is the fusion rules for the quasiparticle types, and various maps of the state spaces, including the adiabatic statistics, and the "twist" for each quasiparticle, which is the phase picked up by the state under a 2π rotation of the quasiparticle. Most of this data can be read off from the conformal blocks, or defined, but the statistics and the twist must be calculated adiabatically. The twist for a quasiparticle of type α is $\theta_\alpha = e^{2\pi i s_\alpha}$, where s_α is its spin. The spin can be calculated adiabatically by transporting a quasihole on a curved surface, and equals the conformal weight of the corresponding field in the CFT whenever the 2D theory is in a massive phase.[18]

With this structure in hand, one can calculate the *quantum dimensions* d_α; d_α is a real number which is defined for each quasiparticle type α.[17] There is an important concept of a *unitary* MTC,[17] which implies that the norm squares of the states are positive, as required in the quantum-mechanical theory, here in 2 + 1 dimensions. In particular, in a unitary MTC $d_\alpha > 0$ for all α. Now the CFT used for the conformal blocks also (if it is rational) corresponds to a MTC, and so quantum dimensions are found there also. According to the above arguments, when the construction gives a topological phase, these two MTCs are equivalent (i.e. fusion rules, braiding/statistics, and spin are the same, and so on), and so the d_α's are the same. But if the CFT is non-unitary (i.e. its inner product in the sense of a 1 + 1 quantum field theory is not positive definite), then in all known examples it contains negative conformal weights, and it can be shown that the presence of these implies that some quantum dimensions are negative.[15] Thus such CFTs cannot produce an acceptable topological phase, and it seems likely that the 2 + 1 system is gapless.[4] This point is of particular interest currently, as some authors have exhibited interesting trial wavefunctions that are conformal blocks in non-unitary rational CFTs.[19,20] Our arguments imply that the corresponding special Hamiltonians for which these wavefunctions are zero-energy eigenstates are probably gapless.

8. Hall Viscosity

A further property of a fluid is what we call *Hall viscosity*.[21] The Hall viscosity is the analog in viscosity of Hall conductivity; it is a non-dissipative transport coefficient which vanishes if time reversal or space reflection symmetries are unbroken. It can be related to an adiabatic transport calculation, and was so calculated for non-

interacting particles[22] and for the filled LLL.[21] In a topological phase obtained from conformal blocks as trial wavefunctions, on using the above arguments, it is proportional to the conformal weight of the particle field.[4] For example, for the Laughlin states it is $\eta^{(A)} = \hbar Q\overline{n}/4$, where $\overline{n}$ is the particle density. In general, it is $\hbar\overline{n}/2$ times the (orbital) spin per particle in the ground state. While not part of the MTC description of a phase, it appears to be a constant throughout a topological phase, like Hall conductivity, as long as translational and rotational symmetry is not broken.

9. Conclusion

The strategy we have used in this work differs from some previous arguments that attempted to calculate directly the adiabatic statistics for some trial state. Instead, our strategy was to determine necessary and sufficient conditions for the result to equal the monodromy. The condition is that the perturbed 2D field theory has to be in a massive phase. It seems likely that it holds in many cases in practice. Proving rigorously that a particular case is in a massive phase is likely to be very difficult in general. However, for the MR case (with the charge sector removed), an explicit calculation at a physical level of rigor (i.e. assuming that screening occurs in a certain plasma) has been done in Ref. 4.

While it does not seem possible to find a conformal block that is a trial ground state for every fractional QH phase, the arguments presented here can be plausibly combined with other techniques (the hierarchy, particle-hole transformation) to obtain results for a much larger class of QH states. Possibly the arguments can be generalized to apply more directly to such states.

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Rapporteur Talk by I. Affleck: One Dimensional Quantum Condensed Matter

1. Introduction

The study of one dimensional (1D) systems plays a special role in the quantum theory of condensed matter. Most of the standard mean field type approximations, that often work reasonably well in general, fail in one dimension. This follows, for example, from Coleman's theorem, or a quantum version of the Mermin-Wagner theorem, which implies that spontaneous breaking of continuous systems is generally impossible even at zero temperature in one dimension. Thus spin wave theory of antiferromanetism is not valid. Quantum fluctuations dominate in one dimension and the behaviour is very exotic and non-standard.

Fortunately, at the same time, a powerful array of theoretical methods are available in one dimension. This includes numerical techniques which are especially powerful in 1D both due to the smaller number of lattice sites needed to extrapolate to the thermodynamic limit and also because the Density Matrix Renormalization Group (DMRG) can be applied with great effect in that case.[1] Field theory methods are very successful in one dimension. This is in large part due to bosonization, which maps certain interacting fermion models into non-interacting bosonic ones. Renormalization group techniques are applied for all dimensions (1,2,3, ...) but seem to be especially useful in 1D where they are very well understood. Conformal invariance at critical points implies the infinite dimensional group of conformal symmetries in 1D, leading to a very complete understanding of many critical points. Certain important 1D models, such as the spin-1/2 antiferromagnetic chain are integrable in 1D, leading to various exact and unusual approximate results.

Actually applying the very extensive 1D theoretical results to experiments has long been a challenge however. This is in large part because most candidate experimental systems consist of weakly coupled chains and even weak inter-chain couplings can completely change the behaviour at low energy scales. In recent decades experimental realizations of 1D models have emerged which don't suffer from this

problem. These include quantum wires and carbon nanotubes.

To some extent the relationship between theory and experiment is inverted in 1D compared to higher dimensional quantum condensed matter. Certain theoretical models are so well understood and predict such exotic behaviour that they have motivated experimentalists to try to find or build experimental realizations. By contrast in higher dimensions the experiments usually precede and motivate the theory. Generally speaking the amount of theoretical results in 1D seem to vastly exceed the number of experimental ones.

Apart from the intrinsic fascination of the theory and the possibility of making connections with experiments, another motivation for studying 1D systems is to provide some inspiration for theories of unusual phenomena in higher dimensions.

In this half-hour report it was impossible to cover even all the interesting and important recent 1D research by participants at this conference. Instead, I just gave a few examples of recent activity which has attracted my interest. I offer my apologies to all my colleagues whose work I didn't have time to summarize. In the next section, I discuss recent progress in going "beyond the Luttinger model" to include the perplexing and important effects of band curvature for 1D interacting electrons. In Sec. III I mention some recent results on quasi-1D antiferromagnets, where the ever-present problem of inter-chain couplings lurks. In Sec. IV I mention the recent success, after several decades of efforts, at finding an experimental realization of a multi-channel Kondo model which exhibits "non Fermi liquid" behaviour. In Sec. V I mention work on stripe phenomena in Hubbard and t-J ladder models, as an example of how 1D methods can be used to attempt to understand a complex phenomenon in higher dimensions.

2. Beyond the Luttinger Model

Enormous progress on understanding interacting electron systems in 1D has been made by keeping only a narrow band of states near the 2 Fermi points, $\pm k_F$ and approximating the dispersion relation as linear. This maps the non-interacting part of the Hamiltonian into a relativistic Dirac model and is the starting point for bosonization.[2] In some cases (eg. with no Umklapp scattering) the interactions leave the bosonized model non-interacting leading to many results which are often assumed to be exact at low energies. Including band curvature in the fermionic model generates formally irrelevant interaction terms[3,4] in the bosonized model involving 3 or more derivatives and powers of the boson field. This approach focusses on the low energy excitations and only gives information about the dynamic structure function, for example, at wave-vectors near 0 and $\pm 2k_F$. Furthermore, it misses certain qualitative features even at these wave-vectors. For instance, it predicts a δ-function for the dynamic structure function at small wave-vector, q:

$$S(q,\omega) \propto |q|\delta(\omega - v|q|). \tag{1}$$

In fact this δ-function should actually be replaced by a peak of width $\propto q^2$ in general, with a highly non-trivial line-shape and a tail stretching to much higher

energies.[5] [See Fig. (1).] The situation at arbitrary q is even more complex. In some cases (eg. translationally invariant models or a tight-binding model at half-filling), there may be a sharp lower bound to the spectral weight which is characterized by a power law singularity.[5] It is possible that there is also be a power-law singularity at a higher frequency, separating the main peak from the high energy tail, in some cases,[5] as in Fig. (1). Such an unusual singularity sitting inside a continuum may require integrability and not occur in generic models.[6] Studying these band curvature effects has long been a difficult open problem. In some cases, the lower edge singularity corresponds to a single hole far below the Fermi energy and a large number of particles and holes very close to $\pm k_F$. As such, it is formally similar to a model of a single heavy particle interacting with a band of low energy excitations. It can be studied by the methods developed for this problem, which were initially developed to study the X-ray edge singularity in metals.[7–9] This technique consists of bosonizing the excitations near $\pm k_F$, but treating the deep hole separately. A unitary transformation then decouples the deep hole from the Fermi surface excitations, up to irrelevant operators. Combining this technique with exact Bethe ansatz results on the finite size spectrum, together with conformal field theory (CFT) results leads to exact predictions for the lower edge exponent *at all* q, for integrable models.[10,11] These are in good agreement with recently developed real time DMRG calculations.[1,10]

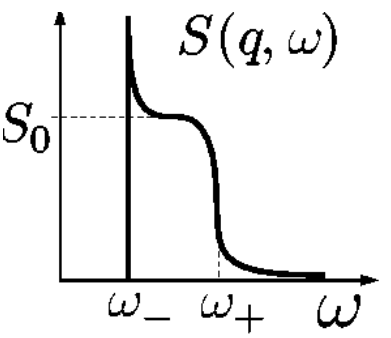

Fig. 1. Schematic plot of the spectral density, $S(q,\omega)$ at small q for interacting fermions in $D = 1$. The width of the main asymmetric peak scales as q^2 but there is also a tail extending to much higher frequencies.

3. Quasi 1D Antiferromagnets

Haldane's prediction[12] of an excitation gap above a singlet ground state to the lowest, spin triplet, excitation for integer spin antiferromagnetic Heisenberg chains, has led to a lot of theoretical and experimental activity over the last 25 years. The gap has now been measured by DMRG and exact diagonalization to 8 significant digits, $\Delta \approx .41047925(4)$ J.[15] The existence of a gap has been well-established by

neutron scattering experiments.[13,14] Precisely because of the gap, the 1D physics is robust against sufficiently weak interchain couplings. The Bose-Einstein Condensation transition at a critical magnetic field where the gap closes has also been confirmed in several experiments[16,17] (although it is smeared by anisotropy). However, other fundamental features of the spectral function have so far evaded experimental confirmation. The theoretical result is shown in Fig. (2a), which shows the region of non-zero spectral weight. The single particle is the minimum energy excitation near $k = \pi$. On the other hand near $k = 0$ the lowest excitation is a 2-particle one, with energy approaching 2Δ as $k \to 0$. The single particle excitation merges into the 2-particle continuum at a critical wave-vector, $k_c \approx .23\pi$ - $.24\pi$. While there are interesting and still controversial[15] issues related to the behaviour near k_c, this physics has so far eluded experimental confirmation, due to the fact that the intensity vanishes quadratically as $k \to 0$. Another fundamental theoretical prediction is that, at $k = \pi$ there is another gap from the single particle up to the extremely weak 3-particle continuum, beginning at 3Δ. Attempts to measure[18] this second gap in $CsNiCl_3$ were stymied due to interchain coupling effects. Confirming these basic features remains a fundamental experimental challenge which may finally be overcome by the increased flux available at the Spallation Neutron Source.

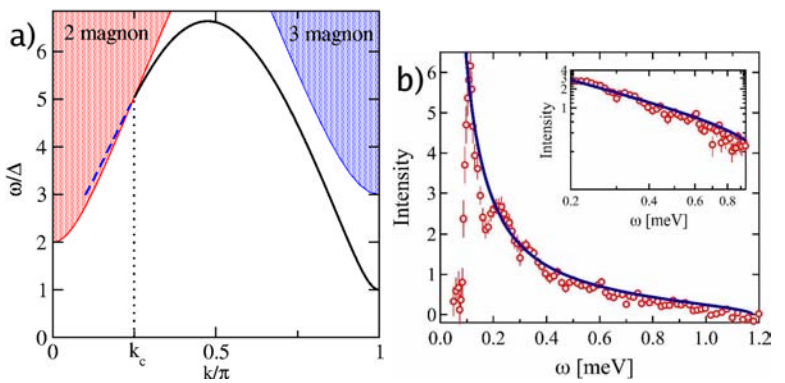

Fig. 2. (colour online) a) Schematic plot of the region of non-zero spectral weight for the $S = 1$ Heisenberg antiferromagnetic chain. The single magnon excitation merges into the 2-magnon continuum at a critical wave-vector $k_c \approx .24\pi$. At $k = \pi$ there is a gap Δ to the single magnon and then another gap 2Δ to the bottom of the 3-magnon continuum. b) Neutron scattering measurement of $S(q, \omega)$ for Cs_2CuCl_4 compared to 1 dimensional 2-spinon result.[24]

Cs_2CuCl_4 is a quasi 1D triangular lattice antiferromagnet with the inter-chain couplings weaker by a modest factor of $\approx .34$ compared to the intra-chain coupling. Neutron scattering experiments by Coldea et al.[19] found magnetic order but very non-spin wave like features at higher energies. This has led to various proposals of exotic two dimensional (2D) phases that were "nearby" in parameter space.[20–23] However, in a triumph of sorts for 1D methods, it was observed[24] that much of

the data can be explained by starting with the 1D 2-spinon excitations and letting them propagate between chains in an RPA or "inter-chain mean field theory"-like approximation. [See Fig. (2b).] An important point is that frustration reduces the effects of the sizeable interchain coupling.

4. Non-Fermi Liquid Kondo Effects

Kondo type Hamiltonians, describing quantum impurity problems with all the interactions at or near one point, can be studied by 1D methods such as an extension of the Bethe ansatz originally due to N. Andrei and P. Weigmann,[25,26] bosonization techniques[27] and boundary conformal field theory,[28] even though experimental realizations are usually 2 or 3 dimensional. Although traditional experimental realizations involve dilute magnetic impurities in a metal, more recently the Kondo effect has been seen in experiments on gated semi-conductor quantum dots connected to 2 dimensional leads.[29–31] While the standard Kondo models exhibit Nozières' "local Fermi liquid" behaviour,[32] below the characteristic Kondo temperature, T_K, other variants with two or more channels (and sometimes more impurity spins) can have non-Fermi liquid ground states due to frustration of the "Kondo screening" of the impurity by the mobile electrons.[28,33–35] These are generally unstable fixed points requiring tuning of parameters to special values. For instance, in the 2-channel spin-1/2 Kondo model, the Kondo couplings of the 2 channels must be equal to realize this quantum critial point. The exact critical behaviour at these quantum critical points (QCP's) is known from these 1D theoretical methods. Claims, over the decades, of "naturally occurring" realizations of these exotic QCP's have been vigorously disputed. Recently a perhaps indisputable experimental quantum dot realization was "tailor-made" in a gated semi-conductor heterostructure by the Goldhaber-Gordon group.[36][See Fig. (3a).] The non-Fermi liquid behaviour was evidenced by the occurance of a critical exponent of 1/2, corresponding to exotic dimension 3/2 boundary operators appearing in the vicinity of the QCP.[28] [See Fig. (3b).] This provides an example, par excellence, of the recent trend towards tuning Hamiltonians governing experimental systems to realize theoretical models of interest. Current theoretical/experimental issues in this field include non-equilibrium transport,[37] inelastic impurity scattering and electron dephasing[38] and the occurance of a large Kondo screening cloud (with size of order v_F/T_K).[39]

5. Stripes in Hubbard and *t-J* ladders

I close with an example of attempting to gain insight about a 2 dimensional phenomena from studying 1 dimensional models. The possible occurance of "stripe" phases in high-T_c superconductors and related compounds and whether this behaviour helps or interferes with superconductivity has long been an open question,[40] brought into sharp focus once again as an interpretation of the small Fermi surface observed in high-field oscillation experiments.[41] Stripes have also been observed in numerical simulations based on the Density Matrix Renormalization Group.[42] At

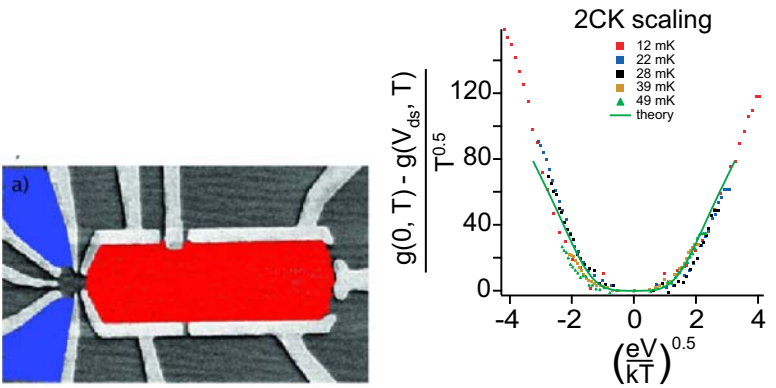

Fig. 3. (colour online) a) Schematic diagram of device used in [36] to observe 2 channel Kondo quantum critical point. The blue regions represent the leads and the red region is a "large dot". b) Difference between zero bias and finite bias conductance[36] divided by $\sqrt{T}$ plotted versus square root of source-drain voltage divided by temperature, exhibiting scaling exponent 1/2, predicted by CFT.

present, this method is primarily useful for 1D or quasi 1D systems (but see the paper by S. White in these proceedings on recent developments.[1]) The largest Hubbard or t-J model systems studied to date are n-leg "ladders" with n as large as 8 and the ladder length much larger. Several DMRG groups have found evidence for "vertical stripes" in these systems, at intermediate dopings around 1/6.[43–45] [See Fig. (4).] These can be regarded as "merely" Friedel oscillations induced by the open boundary conditions in the leg direction, which are the most efficient to use in DMRG. However, further consideration shows that this boundary-induced effect can teach us something about the "bulk" behaviour of these systems, when the number of rungs in the ladder becomes infinite. We then have a 1D system which can be studied by bosonization and RG methods, albeit an increasingly complicated one as the number of legs increases. General Luttinger liquid techniques tell us that the stripe wave-vector of the density oscillations is the same as the wave-vector occuring in the density-density correlations. As far as I know, none of the numerous phases found by bosonization methods prior to 2007 can explain these oscillations in 4-leg ladders at a wave-vector corresponds to 4 holes (1 per leg) per stripe. Recently an exhaustive study of numerous phases that might occur in such systems found the only consistent ones to have no pairing but instead "bipairing".[46] That is to say, there is a gap to add 1 or 2 particles put no gap to add 4. Similarly the correlation functions decay exponentially for pair operators but with a power-law for charge 4 operators. By an approximate mapping of a 4-leg fermionic ladder onto a 2-leg bosonic ladder, we may relate this bipairing to simple boson pairing.[47] It is not clear at this point if this bipairing conjecture is consistent with *all* the DMRG data, including that on the finite-size excitation gaps. It is even less clear how to understand the stripes in 6-leg ladders which exhbit only 4 holes per stripe.[43]

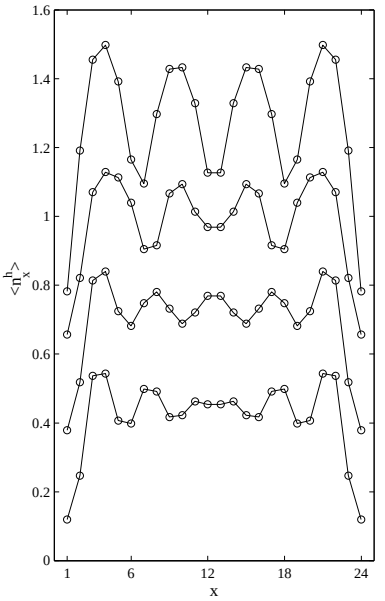

Fig. 4. Hole density per rung from DMRG simulations on 4-leg t-J ladders.[44] For the top curve, corresponding to a hole density of 1/6, we see four "stripes" each containing 4 holes. At lower densities, we see more peaks, not less, indicating that stripes are not forming.

6. Conclusions

The theory of one dimensional quantum condensed matter is thriving with many new results appearing based on the powerful combination of numerical and analytic methods which can be brought to bear on the subject. The large number of low dimensional compounds under study and the precision control of semi-conductor devices is also leading to some impressive convergence of theory and experiment.

Acknowledgments

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Discussion

J. Chalker Questions and comments?

M. Cohen For the finite q work on the Luttinger systems, do you then include umklapps?

I. Affleck Yes. So in fact most of the results I showed you, in fermion language it is a one dimensional chain of spinless fermions with nearest neighbour interactions. This model is equivalent to the XXZ spin chain. And it is exactly integrable. So we can apply all these techniques. In that case, yes, one can include umklap. So there are results for instance at half filling. In that case there is a non oscillating umklap interaction one can include in the Hamiltonian. And in fact we are including that. But it turns out that it is irrelevant. Basically, if the repulsive interactions are weak enough, this umklap interaction is irrelevant. And in fact there is a critical value of the repulsive interaction where it becomes relevant, and a gap develops. What I was talking about was results in the regime where the umklap is irrelevant. So yes, it is being included. It has some effects, but it does not destroy the gap or the singularities.

N. Andrei The question of whether umklap is relevant or not depends on what you want to calculate, on what quantity you want to look at. For instance if you look at conductance, the only way that the system can relax its energy and have finite conductivity is through umklap. So although they are highly irrelevant, sometimes infinitely irrelevant, they are the only source of energy dissipation and you cannot do without them.

I. Affleck Yes I agree completely. In fact one of the open questions here is how to use some of these methods to understand conductivity which has this very delicate dependence on these irrelevant umklap scattering processes.

L. Glazman I want to make a small comment about the edge singularities. So my understanding is that in general integrable models do not allow easily to calculate the dynamic quantities and their dynamic structure factors. But there are

some special models, and one is Cologero-Sutherland, where there is a wavefunction which allows directly to find the edge singularities. I think it is only one model which allows to find the exponents and to check out the results from these later works on this problem with this integrable solution.

N. Andrei More recently one has been able to calculate the edge singularities in other models, in a sense Luttinger liquids, in which you can have edge singularities in and out of equilibrium. And you can calculate the exponents exactly, and even how the exponents are rounded off by finite lifetimes. So it depends very much on which system you look at.

I. Affleck Which system in particular ?

N. Andrei For example impurity models where you have a local interaction between the leads and the dot. That is the model Matveev and Larkin were looking at, where you have the dot interacting with the impurity. And its a charge-charge interaction between the dot and the impurity, there is a Fermi edge singularity which becomes computable, measurable, and you can even now with the exact solution compute how it is destroyed by the widening of finite life effects.

I. Affleck Which is just tunnelling.

N. Andrei Which means that the level itself, because of the tunnelling, does not live for ever, and it widens the singularity. So it is a higher order calculation in the tunnelling coupling; which were neglected before. One could do resummation of all U, but not resummation of all tunnelling effects. And now one can resum the two.

E. Shimshoni I wanted to make one comment on the connection between experiments and theory in this field. There are some experiments on spin chain systems, spin 1/2 chains, where you see that the thermodynamic data fit to the theory like a perfect fit, quantitative fit to specific heat et cetera. However if you look at transport, like thermal transport in those systems, then you need a more elaborate theory to decipher the spin degree of freedom information, and recover the physics of the underlying 1d system. So there is some work of a collaborator of mine, including ...

I. Affleck In some cases the perturbations to integrability do not change things too much, but there always seem to be certain subtle effects that arise from integrability and understanding what those are, and what is generic and what is not, is one of the important open questions.

E. Shimshoni Right. So just wanted to mention that there is some work that has addressed these open questions. And it is useful in explaining experimental data.

B. Halperin I just want to mention in terms of experiments that of course there are a number of experiments in one dimensional metals. In particular, there are beautiful experiments conducted by Amir Yacobi and his coworkers in which they can do momentum conserved tunnelling into a one dimensional wire with cleaved edge overgrowth. Issues of finite curvature are quite relevant there. Of course the hardest part for the experimentalist is actually trying to reach the

lowest energies, which would be the Luttinger liquid regime. But in fact there is a lot of data at high energy and some of them are not so easy to understand. So we really need better theories that take into account deviations from the Luttinger liquid, from linear dispersion, in order to even discuss these experiments.

L. Balents On the subject of interesting experiments in 1d, one that stands out to me, part of the experiments were done by Ong which is an interesting combination of this session and the previous one on very high field transport in bismuth in the ultra quantum limit. In this high field limit you have an electrons basically in the lowest Landau level, but because they have momentum along the field, they are behaving very quasi one dimensionally. They are seeing resistance anisotropies of order 10^6 in that regime. But it is kind of a self organized one dimensional system. There is no particular place for these quasi 1d electrons to be sitting. A lot of interesting physics going on there. A lot of potential for the theorists.

T. Senthil A very general question. To what extent all the weird things that occur in 1d go over to higher dimensions. So I thought of making a couple of comments here. If you think about insulating quantum magnets which, from the spin 1/2 chain in 1d which we understand very well. So that kind of physics I think is not so mature. We can more or less safely say that it seems possible that that kind of physics goes over to 2 dimensions. And there are these critical spin liquid phases that previous people talked about. But for the metallic 1d systems, the Luttinger liquid and so on, as far as I know, no one has ever presented a theory of such a state in two or higher dimensions. Though this was much discussed in the 90's.

E. Shimshoni So I wanted to ask about the 1d version of the Kondo lattice. If you could comment on what has been achieved on this and if it is related to higher dimensional systems like heavy fermions.

I. Affleck I have not thought about this problem for a few years. But, yes, there has been quite a bit of work on 1d Kondo lattices, and I can think of various phases that have been understood using these 1d techniques. There are for instance phases that have a gapless and a charge excitations, there are phases where everything is gapped. In certain regimes there are ferromagnetic phases. So I think that quite a bit is understood about various versions of Kondo lattice models in 1 dimension. In terms of what it teaches about higher dimensions, I think in many ways they are particularities of one dimensional physics, so it is a little bit hard to take any direct lessons.

C. Varma Related to Senthil's remark related to physics in dimension higher than 1 being completely different than one dimension, I wanted to point out that about 6 years ago there is a very lovely paper by Philippe Noziere tracing this problem in a very simple and easily understandable way to the issue of recoil, or in effect Compton scattering. That problem is in fact completely different in one dimension, which is in fact closer to zero dimension – the issue of recoil – than in two or higher dimension.

I. Affleck This will be in the context of these X-ray edge singularities?

J. Chalker OK. So there is one topic that has been on the edge of several sessions including this one, which is the one of topological insulators and edge or surface states. And I thought it would be useful to have a prepared contribution by Alexei Kitaev on that area.

A. Kitaev – prepared comment The ideas described are reviewed in: A. Kitaev, arXiv:0901.2686, *Periodic table for topological insulators and superconductors*, to appear in the Proceedings of the L.D.Landau Memorial Conference *Advances in Theoretical Physics*, June 22-26, 2008.

J. Chalker So, are there questions or comments more generally on topological insulators?

X.-G. Wen One question is, when you classify this mass term, those matrix have a fixed dimension. Right, these matrix have a certain dimension like 5 or 6. These gamma matrices you mentioned.

A. Kitaev No. You may ask the classification question for a fixed dimension. And I do not know the answer. But I am taking this approach to be more of a mathematician than a physicist. I allow this dimension to grow, which means that one can add trivial degrees of freedom to the system. And in this case the answer is rather simple. So I allow this matrix gamma to be arbitrarily big, but satisfying the correct commutation relations.

X.-G. Wen Thank you.

M. Freedman Alexei, I wonder, could you comment on how this K-theoretic approach illuminates certain fractional Hall states, or in more detail how it illuminates the topological insulator model?

A. Kitaev No. I did not say anything. There is indeed a problem how to extend this classification to interacting phases. And there are actually two problems. There are some interacting phases that do not fit into the classification, like fractional quantum Hall phases. And if we just take the phases that are in the table, like this phase. Take any phase. It is a non interacting phase. And it is classified by a Z2 number or a Z number. But we may ask the question whether this classification is stable to interactions, whether it is possible to connect two topologically distinct phases by a path that goes through a strongly interacting phase. And in some cases the answer is no, and in other cases the answer is yes. And there is no general answer so far.

N. Read So for non interacting fermions, now with disorder, there exists a symmetry class classification of random matrices that was completed by Altland and Zirnbauer, and I wondered if you could comment on how this is related to what you are doing?

A. Kitaev That Bott periodicity table lists some homogeneous spaces and these are exactly those spaces that are found in that classification of random matrices. But the random matrix classification applies only in dimension zero. And to make the next step, to dimension one or to dimension two, people have used some special methods, ad hoc methods. In each particular case the method was different. What is important in this table is that we have uniform pattern. A

shift in dimension corresponds to a step up in the table. I mean, in the same symmetry class, when we go from dimension 1 to dimension 2, the topological number changes in a certain way. And that is new, compared to the existing symmetry classification.

X.-G. Wen Another question is, if you have a SU(2) rotation symmetry, like spin, whether you have another table, or that is trivial?

Kitaev This table is specifically restricted to two symmetries, charge conservation and time reversal symmetry. I do not know what happens if you add SU(2) symmetry. But in principle there is a technique in place to answer that question. It is called equivariant K theory.

J. Chalker OK. I am conscious that we cut the discussion of quantum Hall systems short at the end of the first half of this session. I wonder if there are any comments that people would like to make in relation that topic?

M. Freedman Actually it is a comment on quantum Hall and this table, together. I know that the emphasis of your presentation is on non interacting systems. But is the Moore-Read state a special case where using Majorana fermions you could put it into this table essentially where the strontium ruthenate entry occurs.

A. Kitaev Exactly. It is an interacting system. But one can find some structure that is the same as for non interacting systems.

M. Freedman By going from Dirac to Majorana fermions?

A. Kitaev Basically, as soon as we identified fermionic degrees of freedom we can use this technique.

J. Chalker So I believe Duncan had a point to make.

D. Haldane The ideas discussed are published as: arXiv:0707.3637, B. Andrei Bernevig and F. D. M. Haldane, *Fractional Quantum Hall States and Jack Polynomials*, Phys. Rev. Lett. **100**, 246802 (2008).

J. Chalker Comments or questions provoked by that?

A. Auerbach Explain what the Wick theorem like properties are?

D. Haldane The Jack polynomials have this basis; you have a root state that is a kind of occupation number thing. And then all the other configurations within it come about by ones that you can obtain by squeezing things together. It is called dominances of partitions. And depending on the property of the Jack polynomial which contains some physics k=1 for Laughlin, k=2 for Moore-Read, etc... the coefficients of these other terms come in here. And it turns out that if you look at this hierarchy of things you can squeeze there is something like contraction properties where at least some of these coefficients are kind of factorized when I kind of try to squeeze things together. So there are some things which start to be reminiscent of Wick's theorem in these things. Obviously Wick's theorem is the basis of all standard many body calculations based on free bosons or free fermions. And if there is a dream that we can actually find free anyon algebraic methods, underlying that will be some kind of generalization of Wick theorem to this kind of case.

X.-G. Wen I have two questions. One is whether the Jack polynomial is a generator in the sense that some other pattern of 1-0 may also OK. The other question is whether a Jack polynomial is too general in the sense that some of Jack polynomials seem to correspond to non unitary conformal field theory. I am curious what your comments are.

D. Haldane Yes. Some of these things are related to non unitary field theories. And it is a question. What exactly that means is kind of a hot topic at the moment. I do not have an answer to that. Whether they are general enough? In fact this squeezing property, or this dominance thing is perhaps the more general thing. Because the Jacks are an example of that. They satisfy certain differential equations. In fact, for example, one can construct these kind of wave functions. This dominance property is the thing that is the Laughlin quasi particle state. In this case the hole of course is the old thing. The quasi particle is outside the Jack thing because it violates at one time the not more than one particle in three orbitals case. So these things are actually expandable. They are also writable as various sums of Jack and things.

X.G. Wen But do know the most general mathematical structure that gives you the squeezing rule?

D. Haldane No. It is certainly going to be more general. The Jack polynomial is an example of a family of polynomials that have this. But amazingly there is a large number of formulas...there is a literature on Jack polynomials, and we are seeing that a lot of things in conformal field theory can come right out of those formulas, like identifying the conformal weights of the primary field operators and things. So there was some other body of mathematics that turns out to contain a lot of this, and may be the key to doing some more things.

J. Chalker So for the final few minutes of the session, I thought it might be useful to try to make a bridge with this afternoon and to talk about cold atom realizations of one dimensional systems and Eugene Demler is going to introduce that topic.

E. Demler – prepared comment

J. Chalker The general discussion of cold atomic systems will be left until this afternoon.

Session 5

Systems of Ultra-Cold Atoms, and Advanced Computational Methods

Chair: *Peter Zoller*, University of Innsbruck, Austria
Rapporteurs: *Immanuel Bloch*, Johannes Gutenberg-Universität, Germany and *Steven R. White*, University of California Irvine, USA
Scientific secretary: *Thomas Durt* (Vrije Universiteit Brussel)

Rapporteur talk by I. Bloch: Strongly Correlated Quantum Systems with Ultracold Atoms in Optical Lattices

1. Abstract

This article summarizes work on strongly correlated quantum many-body phases with ultracold atoms in artificial crystals formed by laser light – so called 'optical lattices'. The generation of such optical potentials, typical detection methods and some realizations of strongly correlated quantum phases are discussed and an outlook on the state-of-the-art in the field is given.

2. Introduction

Ultracold quantum gases in optical lattices form almost ideal conditions to analyze the physics of strongly correlated quantum phases in periodic potentials.[1–3] Such strongly correlated quantum phases are of fundamental interest in condensed matter physics, as they lie at the heart of topical quantum materials, such as high-T_c superconductors and quantum magnets, which pose a challenge to our basic understanding of interacting many-body systems. Quite generally, such strongly interacting quantum phases arise, when the interaction energy between two particles

dominates over the kinetic energy of the two particles. Such a regime can either be achieved by increasing the interaction strength between the atoms via Feshbach resonances,[4,5] or by decreasing the kinetic energy, such that eventually the interaction energy is the largest energy scale in the system. The latter can for example be achieved by increasing the optical lattice depth.

This article gives a brief summary of the field of optical lattices and the physics of strongly interacting quantum phases with ultracold atoms in such optical lattices. A prominent example hereof are the bosonic[6–11] or fermionic[12,13] Mott insulators and the formation of 1D fermionized Tonks-Girardeau gases,[14,15] which have been realized with ultracold atoms.[16,17] Another major field of research with ultracold atoms has been reached in experiments that explore the crossover from a molecular BEC to a BCS-superfluid of weakly bound Cooper-pairs.[18–21] These are discussed in the contribution of W. Ketterle in this volume. As an outlook, novel systems and novel detection techniques used in the field are discussed.

3. Optical Lattices

3.1. *Optical dipole force*

In the interaction of atoms with coherent light fields, two fundamental forces arise.[22,23] The so called Doppler force is dissipative in nature an can be used to efficiently laser cool a gas of atoms and relies on the radiation pressure together with spontaneous emission. The so called dipole force on the other hand creates a purely conservative potential in which the atoms can move. No cooling can be realized with this dipole force, however if the atoms are cold enough initially, they may be trapped in such a purely optical potential.[24,25] For large detunings of the applied laser field relative to an atomic transition frequency, the potential depth is proportional to the laser intensity $I(\mathbf{r})$ and can be both repulsive or attractive depending on the sign of the detuning.

3.2. *Optical lattice potentials*

A periodic potential can be formed from light fields by overlapping two counter-propagating laser beams. Due to the interference between the two laser beams an optical standing wave with period $\lambda/2$ is formed, in which the atoms can be trapped. By interfering more laser beams, one can obtain one-, two and three-dimensional periodic potentials. Note that by choosing different angles under which the laser beams overlap and interfere, one can also realize periodic potentials with larger periods and different lattice geometries.[26]

By overlapping several of such standing wave configurations, the dimensionality of the periodic potential can be easily varied. For example, a single standing wave forms an array of two-dimensional quantum gases, two orthogonal standing waves result in an array of one-dimensional quantum gases and three overlapping standing waves give rise to a full three-dimensional lattice structure (see Fig. 1).

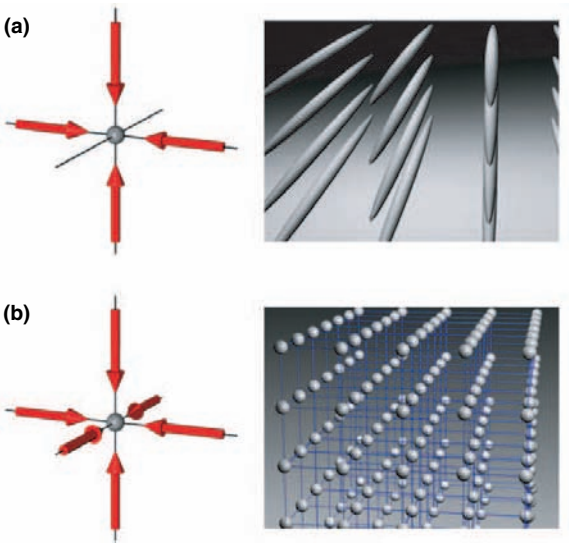

Fig. 1. Two Dimensional **(a)** and three-dimensional **(b)** optical lattice potentials formed by superimposing two or three orthogonal standing waves. For a two-dimensional optical lattice, the atoms are confined to an array of tightly confining one-dimensional potential tubes, whereas in the three-dimensional case the optical lattice can be approximated by a three dimensional simple cubic array of tightly confining harmonic oscillator potentials at each lattice site.

For sufficiently deep optical lattice potentials, the confinement on a single lattice site is also approximately harmonic. Here the atoms are very tightly confined with typical trapping frequencies of up to $\omega_{lat} \simeq 2\pi \times 100\,\mathrm{kHz}$. A natural energy scale for the lattice depths is given by the photon recoil energy $E_r = h^2/(2m\lambda^2)$, with λ denoting the wavelength of laser light forming the optical standing wave.

One important aspect in experiments with ultracold quantum gases, is that in addition to the periodic potential, in general a weak overall harmonic confinement is present due to the transverse gaussian intensity profile of the laser beams. This in general makes trapping of atoms in optical lattices inhomogeneous and breaks the translational symmetry of the problem. Most often, a local density approximation can however be used in order to account for these inhomogeneities. They typically result in having several quantum phases present at once in the atomic traps.

4. Detection Methods

Ultracold atoms offer analogous and complementary detection methods to the ones commonly found in condensed matter physics. Bragg scattering gives access to the dynamical structure factor of the system,[27] as does neutron scattering in condensed matter physics. An analogue of photoemission spectroscopy has recently enabled one to measure the single particle excitation spectrum in a strongly interacting gas

of fermions.[28] Critical velocities of bosonic and fermionic superfluids can be measured by an onset of heating when artificial light structures are 'dragged' through the quantum gases.[29–31] Collective excitations can be revealed using different spectroscopic techniques and by observing the time evolution of the quantum gas in a trap after quenching the system into a non-equilibrum state. By far one of the most commonly used techniques in optical lattice based experiments, is time-of-flight expansion of a quantum gas from the trapping potential. Turning off the lattice potential gives rise to the momentum distribution, whereas turning it off adiabatically can reveal the quasi-momentum distribution of the system. The two methods are briefly discussed below.

4.1. *Time-of-flight and adiabatic mapping*

Sudden release When releasing ultracold quantum gases from an optical lattice, two possible release methods can be chosen. If the lattice potential is turned off abruptly and neglecting any interaction effects, a given Bloch state with quasi-momentum q will expand according to its momentum distribution as a superposition of plane waves with momenta $p_n = \hbar q \pm n \times 2\hbar k$. This is a direct consequence of the fact that Bloch waves can be expressed as a superposition of plane wave states with a discrete set of momenta q and additional integer multiples of the fundamental reciprocal lattice vector $2k$ in a simple cubic lattice with lattice spacing $d = \pi/k$. After a certain time-of-flight time, this momentum distribution can be imaged using standard absorption imaging methods. If only a single Bloch state is populated, as is the case for a Bose-Einstein condensate with quasi-momentum $q = 0$, this results in a series of interference maxima that can be observed after a time-of-flight period t (see Fig. 2). The density distribution observed after a fixed time-of-flight at position $\mathbf{x}$, is nothing but the momentum distribution of the particles trapped in the lattice

$$n(\mathbf{x}) = \left(\frac{M}{\hbar t}\right)^3 |\tilde{w}(\mathbf{k})|^2 \mathcal{G}(\mathbf{k}). \tag{1}$$

Here $\mathbf{k}$ is related to $\mathbf{x}$ by $\mathbf{k} = M\mathbf{x}/\hbar t$ due to the assumption of ballistic expansion while $\tilde{w}(\mathbf{k})$ is the Fourier transform of the on-site Wannier function. The coherence properties of the many-body state are characterized by the Fourier transform

$$\mathcal{G}(\mathbf{k}) = \sum_{\mathbf{R},\mathbf{R}'} e^{i\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')} G^{(1)}(\mathbf{R},\mathbf{R}') \tag{2}$$

of the one-particle density matrix $G^{(1)}(\mathbf{R},\mathbf{R}') = \langle \hat{a}^\dagger_{\mathbf{R}} \hat{a}_{\mathbf{R}'} \rangle$. Here $\hat{a}^\dagger_{\mathbf{R}}(\hat{a}_{\mathbf{R}})$ denotes the particle creation (annihilation) operator at lattice site $\mathbf{R}$.

In a BEC, the long range order in the amplitudes leads to a constant value of the first order coherence function $G^{(1)}(\mathbf{R},\mathbf{R}')$ at large separations $|\mathbf{R} - \mathbf{R}'|$. The resulting momentum distribution results in the standard multiple wave interference pattern obtained with light diffracting off a material grating (see Fig. 2c). The

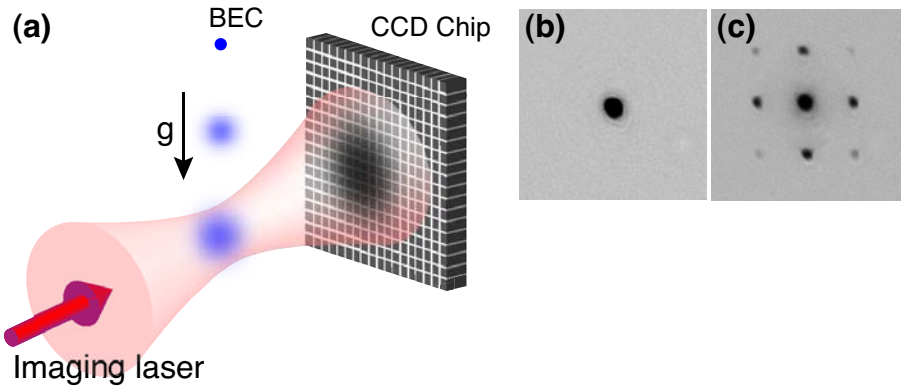

Fig. 2. Schematic setup for absorption imaging after a time-of-flight period **(a)**. Absorption image for a BEC released from a harmonic trap **(b)**. Absorption image for a BEC released from a shallow optical lattice ($V_0 = 6\,E_r$)**(c)**. Note the clearly visible interference peaks in the image.

atomic density distribution observed after a fixed time-of-flight time, thus yields information on the coherence properties of the many-body system.

Adiabatic mapping One of the advantages of using optical lattice potentials is that the lattice depth can be dynamically controlled by simply tuning the laser power. This opens another possibility for releasing the atoms from the lattice potential e.g. by adiabatically converting a deep optical lattice into a shallow one and eventually completely turning off the lattice potential. Under adiabatic transformation of the lattice depth, the quasi-momentum $\mathbf{q}$ is preserved and during the turn off process a Bloch wave in the nth energy band is mapped onto a corresponding free particle momentum $\mathbf{p}$ in the nth Brillouin zone (see Fig. 3).[32–34]

Such a behavior has indeed been observed with both bosonic[33] and fermionic[34] atoms. For the situation of a homogeneously filled lowest energy band, an adiabatic ramp down of the lattice potential leaves the central Brillouin zone – a square of width $2\hbar k$ – fully occupied (see Fig. 4b). If on the other hand higher energy bands are populated, one also observes populations in higher Brillouin zones (see Fig. 4c). As in this method each Bloch wave is mapped onto a specific free-particle momentum state, it can be used to efficiently probe the distribution of the particles over Bloch states in different energy bands.

4.2. *Detection of correlations*

In order to probe interacting many-body quantum states with strong correlations, it is essential to use detection methods that are sensitive to higher order corre-

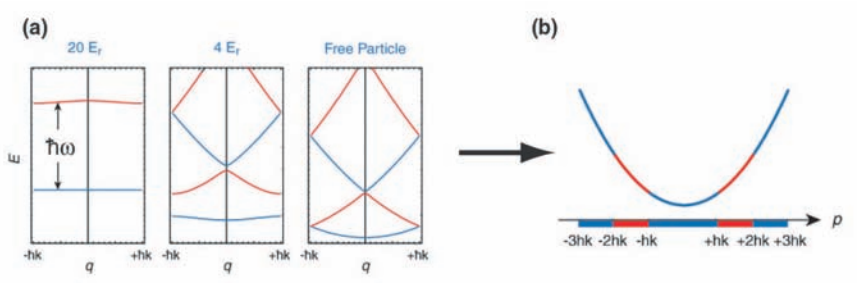

Fig. 3. **(a)** Bloch bands for different potential depths. During an adiabatic ramp down the quasi momentum is conserved and **(b)** a Bloch wave with quasi momentum q in the nth energy band is mapped onto a free particle with momentum p in the nth Brillouin zone of the lattice. Reprinted with permission from Ref. 33.

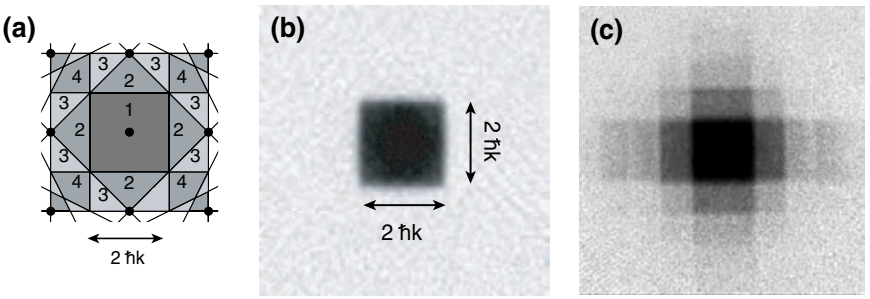

Fig. 4. **(a)** Brillouin zones of a 2D simple cubic optical lattice. For a homogeneously filled lowest Bloch band, an adiabatic shut off of the lattice potential leads to a homogeneously populated first Brillouin zone, which can be observed through absorption imaging after a time-of-flight expansion **(b)**. If in addition higher Bloch bands were populated, higher Brillouin zones become populated as well **(c)**. Reprinted with permission from Ref. 33.

lations. Here, recent proposals for using analogues of quantum optical detection techniques have proven to be novel tools for analyzing strongly interacting quantum matter.[35–40] Most of these techniques make use of the fact that the quantum fluctuations in many observables, such as e.g. the visibility of the interference pattern between two released quantum gases or the fluctuations in the momentum distribution after release from the trap, contain information of the initial correlated quantum state. Whereas in the usual time-of-flight momentum distributions one essentially probes first order coherence properties of the system, the noise-correlation techniques introduced below will yield information on the second (or higher) order correlation properties and therefore possible long range order in real space. Such correlation techniques in expanding atom clouds have begun to be successfully employed in recent experiments, probing the momentum correlations between atomic fragments emerging from a dissociated molecule,[41] revealing the quantum statistics and ordering of bosonic or fermionic atoms in an optical lattice,[42,43] or in explor-

ing the Berezinskii-Kosterlitz-Thouless transition in two-dimensional Bose-Einstein condensates.[44] All the correlation techniques for strongly correlated quantum gases can also greatly benefit from efficient single atom detectors that have recently begun to be used in the context of cold quantum gases.[45–47]

Let us now turn to the observation of density-density correlations in the expanding atom clouds. These are characterized by the density-density correlation function

$$\langle \hat{n}(\mathbf{x})\hat{n}(\mathbf{x}')\rangle = \langle \hat{n}(\mathbf{x})\rangle\langle \hat{n}(\mathbf{x}')\rangle g^{(2)}(\mathbf{x},\mathbf{x}') + \delta(\mathbf{x}-\mathbf{x}')\langle \hat{n}(\mathbf{x})\rangle \tag{3}$$

which contains the normalized pair distribution $g^{(2)}(\mathbf{x},\mathbf{x}')$ and a self correlation term. Relating the operators after time-of-flight expansion to the in-trap momentum operators as in Eq. 1, one obtains:

$$\begin{aligned}\langle \hat{n}_{3D}(\mathbf{x})\hat{n}_{3D}(\mathbf{x}')\rangle_{\mathrm{tof}} &\approx \langle \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k}')\hat{a}(\mathbf{k}')\rangle_{\mathrm{trap}} = \\ &\langle \hat{a}^\dagger(\mathbf{k})\hat{a}^\dagger(\mathbf{k}')\hat{a}(\mathbf{k}')\hat{a}(\mathbf{k})\rangle_{\mathrm{trap}} + \delta_{\mathbf{k}\mathbf{k}'}\langle \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})\rangle_{\mathrm{trap}} \quad .\end{aligned} \tag{4}$$

The first term shows that for $\mathbf{x} \neq \mathbf{x}'$ subtle momentum-momentum correlations of the in-trap quantum states are present in the noise-correlation signal of the expanding atom clouds. For example for the case of a bosonic Mott insulator, deep in the insulating regime ($U/J \to \infty$), one obtains[3] (up to an overall envelope function):

$$\langle \hat{n}_{3D}(\mathbf{x})\hat{n}_{3D}(\mathbf{x}')\rangle \propto 1 + \frac{1}{N^2}\left|\sum_{\mathbf{R}} e^{i(\mathbf{x}-\mathbf{x}')\mathbf{R}\cdot(M/\hbar t)} n_{\mathbf{R}}\right|^2 . \tag{5}$$

Here N denotes the total number of particles, M the mass of a single atom and t the time of flight time.

The above result shows that correlations in the density-density expectation value appear for bosons, whenever the difference $\mathbf{k}-\mathbf{k}'$ is equal to a reciprocal lattice vector $\mathbf{G}$ of the underlying lattice. In real space, where the images are actually taken, this corresponds to special separations for which

$$|\mathbf{x}-\mathbf{x}'| = \ell = \frac{2ht}{\lambda M}. \tag{6}$$

Such spatial correlations or anticorrelations in the quantum noise of the density distribution of expanding atom clouds can in fact be traced back to the famous Hanbury-Brown & Twiss effect[48–50] and its analogue for fermionic particles.[43,47,51–54] For the case of two atoms localized at two lattice sites this can be readily understood in the following way: there are two possible ways for the particles to reach two detectors at positions $\mathbf{x}$ and $\mathbf{x}'$ which differ by exchange. A constructive interference for the case of bosons or a destructive interference for the case of fermions then leads to correlated or anticorrelated quantum fluctuations that are registered in the density-density correlation function.[35] The correlations for the case of a bosonic Mott insulting state and anticorrelations for the case of a fermionic band insulating state have recently been observed experimentally[10,42,43,55] (see also Fig. 5).

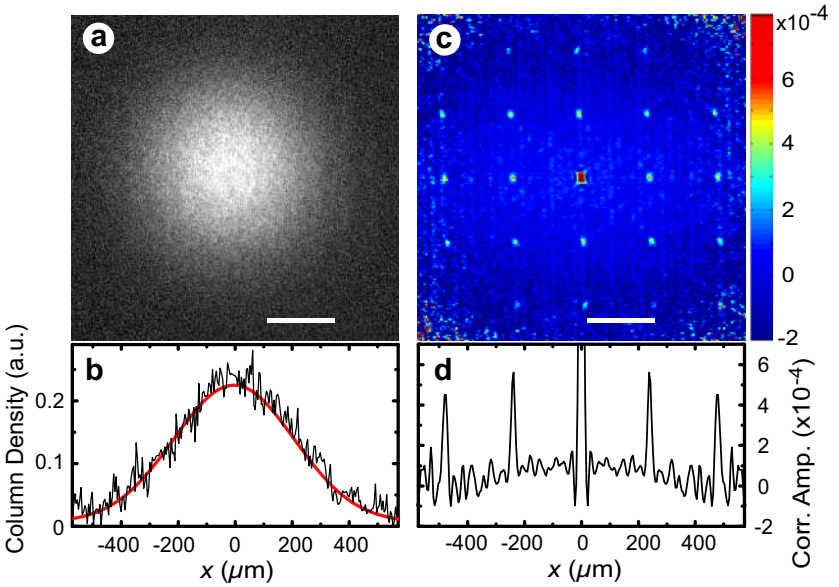

Fig. 5. Noise correlations of a Mott insulator released from a 3D optical lattice. **(a)** Single shot absorption image of a Mott insulator released from an optical lattice and associated cut through the image **(b)**. A statistical correlation analysis over several independent images such as the one in **(a)** yields the correlation function **(c)**. A cut through this two-dimensional correlation function reveals a Hanbury-Brown & Twiss type bunching of the bosonic atoms **(d)**. Reprinted with permission from Ref. 42.

5. Realizations of Strongly Correlated Phases in Optical Lattices

5.1. *Hubbard physics*

The physics of strong correlations in ultracold quantum gases was started by noticing that Hubbard-type models could be exactly realized with ultracold quantum gases in optical lattices.[7] For the case of fermionic particles, such a Hubbard Hamiltonian can be expressed as:

$$\hat{H} = -J \sum_{\langle i,j\rangle,\sigma} \hat{c}^{\dagger}_{i,\sigma}\hat{c}_{j,\sigma} + U \sum_{i} \hat{n}_{i,\downarrow}\hat{n}_{i,\uparrow} + V_t \sum_{i} (i_x^2 + i_y^2 + \gamma^2 i_z^2)\,(\hat{n}_{i,\downarrow} + \hat{n}_{i,\uparrow})\,. \tag{7}$$

Here the indices i, j denote different lattice sites in the three-dimensional system ($i = (i_x, i_y, i_z)$), $\langle i, j\rangle$ neighboring lattice sites, $\sigma \in \{\downarrow, \uparrow\}$ the two different spin states, J the tunneling matrix element and U the effective on-site interaction. The operators $\hat{c}_{i,\sigma}(\hat{c}^{\dagger}_{i,\sigma})$ correspond to the annihilation (creation) operators of a fermion in spin

state σ on the ith lattice site and $\hat{n}_{i,\sigma}$ counts the number of corresponding atoms on the ith lattice site. The strength of the overall harmonic confinement as introduced above is parameterized by the energy offset between two adjacent lattice sites at the trap center $V_t = \frac{1}{2}m\omega_\perp^2 d^2$, with $\omega_\perp = \omega_x = \omega_y \neq \omega_z$ being the horizontal trap frequency and d the lattice constant. The onsite interaction U is directly proportional to the scattering length a between the particles, which is widely tunable via scattering (Feshbach) resonances.[4,5]

For the case of bosonic particles, Fisher et al.[6] noted that such a system should undergo a transition from a superfluid state to a Mott insulating state as the interactions U begin to dominate over the kinetic energy J of the particles. In the context of ultracold atoms, Jaksch et al.[7] pointed out that indeed such a transition should be observable with ultracold bosonic atoms in periodic potentials. A few years after the original proposal, this transition was observed experimentally[8] and marked the start of strong correlation physics with ultracold atoms. So far, several of the characteristic ground state properties of the system in the superfluid or Mott insulating state have been measured.[8–11,31,55–59] One prominent feature is the loss of long-range phase coherence, as the system evolves from a weakly interacting superfluid into a Mott insulator deep in the insulating regime[8,11,56,60] and the corresponding change in number statistics from coherent states to Fock states.[57,59,61]

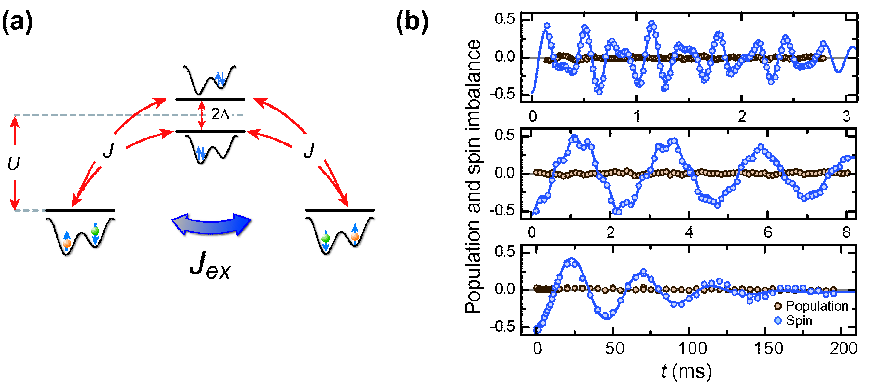

Fig. 6. Superexchange coupling between atoms on neighbouring lattice sites. Virtual hopping processes mediate an effective spin-spin interaction between the atoms, which can be controlled in magnitude and sign using a potential bias between the wells **(a)**. The effective spin-spin interaction emerges when increasing the interaction between the particles relative to their kinetic energy **(b)** (top to bottom) and can be observed in the time evolution of the magnetization dynamics in the double well.[62]

Only recently has it become possible to reveal a Mott insulating state for an equal spin-mixture of repulsively interacting fermions in an optical lattice.[12,13] This was achieved by observing a suppressed pair fraction as the system becomes increasingly strongly interacting,[12,13] or by monitoring the response of the cloud size to an external compression,[13] which can directly reveal the compressibility of the

quantum gas. These experiments have allowed one to carry out ab-initio tests of state-of-the-art numerical methods for strongly correlated electronic systems such as Dynamical Mean Field Theory (DMFT).[13,63,64] For temperatures lower that the superexchange coupling, one expects such a fermionic spin mixture to show an antiferromagnetically ordered phase. Current experiments are still carried out at too high temperatures to observe this phase, however the lowest entropies reached in the system so far seem to be only a factor 2-3 above the ones required for the formation of such an antiferromagnet.[65] It is clear, however, that in order to investigate the low-energy sector of the Hubbard model and its possible connection to high temperature superconductivity[66] in the context of cold atoms,[67] novel cooling methods are needed, as the required temperatures to observe a *d*-wave superconducting phase in real material are typically found to be two orders of magnitude below the superexchange coupling energy scale. Controllable superexchange interactions between neighbouring atoms in an array of double well potentials have however been recently realized, showing how superexchange interactions emerge from the Hubbard model for increasing interactions between the particles[62,68] (see Fig. 6). In general, ultracold atoms allow one to widely tune the coupling terms of spin Hamiltonians by using spin dependent tunnel couplings or by varying the interactions between the atoms.[69,70]

5.1.1. *Low-Dimensional quantum systems*

Optical lattices offer the possibility to change the dimensionality of the underlying potentials. For example, by increasing the lattice depth along one direction in a 3D optical lattice (and thereby decreasing the tunnel coupling in this direction), the system be effectively reduced to a stack of independent two-dimensional lattices. Alternatively, a 1D standing wave creates an array of two-dimensional quantum gases, whereas for a purely two-dimensional optical lattice, tightly confining one-dimensional potential tubes can be realized (see Fig. 1b). For the first case, the Berezinskii-Kosterlitz-Thouless transition has been observed recently,[44] while in the latter case, tightly confined one-dimensional bosonic quntum gases have allowed one to reach the regime of fermionized bosons in a Tonks-Girardeau gas[14–17] and essentially realize bosonic Luttinger liquids.[71]

For the one-dimensional quantum gases at a given 1D density $n_1 = N/L$, the strength of the interactions is characterized by a single dimensionless parameter

$$\gamma = \frac{g_1 n_1}{\hbar^2 n_1^2/M} = \frac{2}{n_1|a_1|}. \tag{8}$$

In marked contrast to the 3D situation, the dimensionless interaction strength γ scales inversely with the 1D density n_1.[15] In one dimension, therefore it is the *low* density limit where interactions dominate.

In the experiments, the Tonks-Girardeau regime could be identified through different observables. On the one hand the axial momentum distribution allows one to reveal a Tonks-Girardeau gas, as it differs both from that of a weakly interacting

BEC and from the one of non-interacting Fermi gas[16] (see Fig. 7a), on the other hand the saturation of the axial expansion energy has been observed for increasing interactions between the particles.[17] As the system becomes more strongly interacting, the atomic wavefunctions tend to separate, which in turn leads to a vanishing probability to detect two atoms at the same position, characterized by the second order correlation function at zero interparticle distance $g^{(2)}(0)$ (see Fig. 7b). This suppression of $g^{(2)}(0)$ with increasing interactions has been detected experimentally[72] (see also ref.[73]) in good agreement with theory.[74]

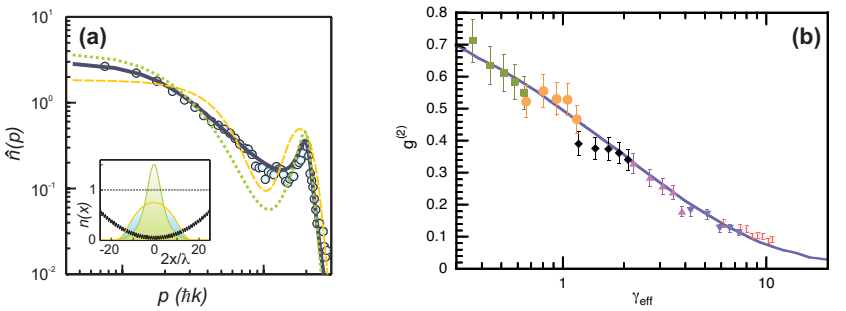

Fig. 7. **(a)** Axial momentum distribution of a lattice based one-dimensional bosonic quantum gas for $\gamma_L \approx 14$. The solid curve is the theoretical momentum distribution based on fermionization and the short and long-dashed curves denote the expected values for a non-interacting Bose gas and a non-interacting Fermi gas, respectively. The insets show the corresponding in-trap density distributions.[16] **(b)** Local pair correlation function of a 1D Bose gas from photo-association measurements as a function of the interaction parameter γ_{eff} averaged over an ensemble of 1D Bose gases.[72] The theoretical prediction is shown as a solid line.

6. Outlook: Novel Systems and Detection Methods

Due to the highly controllable interactions, the ability to shape the underlying periodic potential in almost arbitrary ways, the availability of multiple spin components and the possibility to interface the ultracold quantum gases with other quantum systems, ultracold atoms in optical lattices offer a wide range of research possibilities, including topics such as:

- **Non-equilibrium dynamics** of strongly interacting quantum systems[75–77]
- **Bose-Fermi mixtures** (new composite particles, formation of supersolids or alternating Mott insulators and charge density waves[78,79])
- **Three component systems** (trion formation and analogies to QCD[80,81])
- **Polar molecules** (efficient molecule formation and implementation of spin hamiltonians[82] and long range interactions)

- **Rydberg atoms** (mediation of long range interactions and entanglement generation)
- **Novel exotic many-body phases** (e.g. topological quantum states,[83–86] η-paired superfluid states[87])
- **(Driven) dissipative systems**[88,89] (engineering of coupling to a dissipative system to 'drive' the many-body system into the desired quantum state)
- **Interaction and disorder effects** in periodic potentials (interplay of 'Mott vs. Anderson' physics, (see e.g. ref.[90]))
- **Phonons** (in self-organized dipolar crystal structures,[91] in coupling to a reservoir)
- **Hybrid systems** (e.g. coupling of atomic or molecular quantum gases to nanomechanical oscillators or superconducting qubits)
- **Single site and single atom addressing** in an optical lattice[92,93]

A discussion of each of these topics is beyond the scope of the report, however, below a few of the topics are discussed in more detail.

6.1. *Novel systems*

Towards Quantum Magnetism. Control over the effective spin-spin interactions between neighboring atoms could open up a new avenue for the simulation of quantum magnetism with cold atoms or molecules. Both atoms and molecules offer the ability to implement arbitrary spin Hamiltonians on a lattice.[2] For atoms, the spin-spin interactions are generated via superexchange couplings,[62,69,70] whereas for ultracold molecules the electric dipole-dipole interaction can mediate even stronger spin couplings between individual molecules on neighboring sites.[82] The flexibility in tuning the coupling between molecules allows one to realize various spin-models in these systems. Polar heteronuclear RbK molecules have recently been formed out of a degenerate quantum gas mixture of bosonic ^{87}Rb and fermionic ^{40}K atoms. Feshbach ramps have allowed one to first bind the two atoms into molecules in highly excited ro-vibrational states.[94] A stimulated Raman process has then been used to coherently transfer the molecule population into the ro-vibrational ground state, thus creating a degenerate gas of polar molecules.[95] Initial measurements indicate that the collisional lifetime of these molecules could be long enough to load them into a lattice potential, where such loss processes should be finally suppressed.

Long Ranged Interactions. Interactions between neutral atoms are generally short-ranged and in a lattice are typically restricted to on-site interactions within the lowest energy band of the periodic potential. Longer ranged interactions could open the door to the observation of novel quantum many-body phases and have shown to be useful in the context of quantum information for the generation of entanglement[96,97] or the controlled interaction of distant particles in a lattice. Two routes are currently pursued in this direction: 1) use of polar ground state molecules and long range electric dipole-dipole interactions between the molecules as outlined above

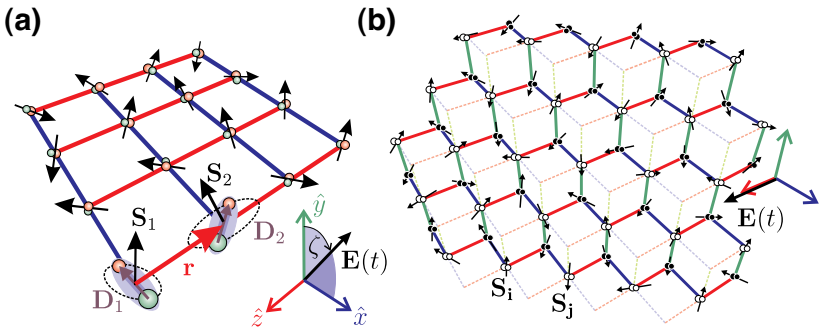

Fig. 8. Spin-spin interactions with polar molecules in optical lattices (see ref.[82]). **(a)** Square lattice in 2D with nearest-neighbour orientation-dependent Ising interactions along the x- and z-directions. Effective interactions between the spins $\mathbf{S}_1$ and $\mathbf{S}_2$ of the molecules in their ro-vibrational ground states are generated with a microwave field $\mathbf{E}(t)$ inducing dipole–dipole interactions between the molecules with dipole moments $\mathbf{D}_1$ and $\mathbf{D}_2$, respectively. **(b)** Two staggered triangular lattices with nearest neighbours oriented along orthogonal triads. The interactions depend on the orientation of the links with respect to the electric field. (Reprinted with permission from A. Micheli and P. Zoller.)

or 2) the use of Rydberg atoms in order to mediate such long range interactions. Because of their large size $\propto n^2 a_0$, Rydberg atoms are highly susceptible to electric fields which allow one to induce large eletric dipole moments within an atom. Even without an applied external electric field the van der Waals interaction between two Rydberg atoms at a distance r can become rather large and long ranged:

$$V_{\text{Rydberg}} \propto \frac{C_6}{r^6}, \tag{9}$$

with the van der Waals coefficient C_6 being proportional to n^{11}, where n denotes the principle quantum number of the Rydberg atom. When trying to excite two atoms simultaneously to a Rydberg state, one generally finds that the strong interaction energy V_{Rydberg} can shift the doubly excited state out of resonance, allowing one only to excite a single atom. This so called dipole blockade mechanism[96,97] has been observed recently in ensembles of (quantum degenerate) ultracold atoms[98–100] and in individual atom pairs.[101,102] Such strongly interacting Rydberg atoms could possibly also lead to novel quantum critical behaviour in the transition from a paramagnetic to a crystalline phase.[103]

6.2. *Novel detection methods*

The ability to image single atoms on single lattice sites offers both novel prospects for the detection and manipulation of ultracold quantum gases in optical lattices. For a counterpropagating laser beam configuration forming a short spaced optical

standing wave with a typical lattice spacing of half of a micrometer or less, this requires a demanding optical microscope in order to detect the atoms via fluorescence imaging. For larger spaced lattices, it has been already been possible to reveal individual atoms on different lattice sites loaded from a laser cooled cloud of atoms[92] (see Fig. 9a). The use of a scanning electron microscope has allowed one to overcome the optical resolution limit and has enabled the imaging of a Bose-Einstein condensate (BEC) loaded into one- and two-dimensional lattice potential[93] (see Fig. 9b). The focussed electron beam locally ionizes the neutral atoms by electron impact and the resulting ions are detected via an ion detector, yielding information on the local neutral atom density. By scanning the electron beam, a two dimensional image of the cloud can subsequently be recorded. So far, however, the technique has not yet achieved single atom sensitivity. However, the excellent spatial resolution of $< 200\,\mathrm{nm}$ achieved in these experiments should allow one to map out the in trap quantum phases with high precision. Furthermore, one could envisage the electron beam to rapidly switch between two locations to record even ordered time and spatial correlation functions of the quantum system. If indeed eventually single atom and single site addressability would become available in short spaced optical lattices, one would be able to observe and control a spin system in two-dimensions with 10000 particles simultaneously in view. Observing dynamical evolutions in these systems, probing their spatial correlations and implementing quantum information processing in such a truly large scale system would offer exciting prospects for future research.

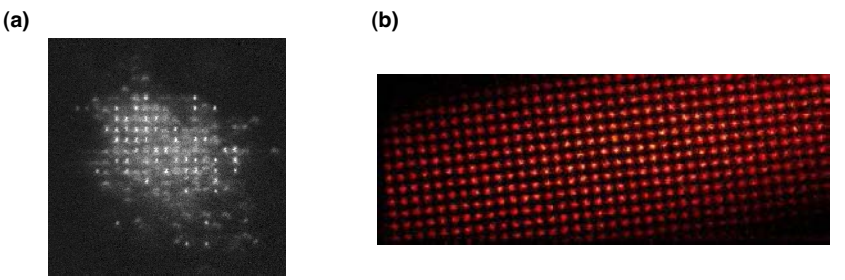

Fig. 9. Single site imaging in optical lattices. **(a)** Image of the flourescence of single trapped atoms in a large spaced optical lattice ($d \approx 5\,\mu\mathrm{m}$) using a high resolution optical microscope objective. **(b)** Electron microscope image (averaged over 300 single images) of ultracold atoms in a 2D optical lattice with a short period of ($d \approx 600\,\mathrm{nm}$) (Reprinted with permission from (a) D. Weiss, Pennstate University and (b) H. Ott, University of Mainz).

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Discussion

P. Zoller Thank you Immanuel, we have time for discussion.

M. Fisher Have there been experiments which measure the superfluid density in a Bose-lattice system by rotating or moving the lattice trying to see whether the atoms are moved along or not?

I. Bloch It is difficult and problematic to measure the superfluid density but one can measure the critical velocity of the superfluid. In the weakly interacting regime both quantities condensate; density and superfluid density are the same thing but in the strongly interacting regime, it is difficult to separate them. Wolfgang could you comment on this?

W. Ketterle In the regime where we worked we did not and could not separate between condensate density and superfluid density. On the other hand we addressed quantum depletion which is the difference between the two.

B. Halperin To preclude the possibility of semantic confusion, let me say that as I understand it, the quantum depletion is defined as the difference between the condensate density and the total density at zero temperature, and of course, it is a signature of interaction effects. For a Galilean invariant system at T=0, the superfluid density is equal to the total density. However, for T not equal to zero, or in the case where there is an optical lattice, the superfluid density is generally different from both the total density and the condensate density. These are three different quantities.

S. Das Sarma My question is related. What is the status of actually experimentally obtaining the phase diagram for the Bose-Hubbard model. In the inhomogeneous system, there has been a lot of work in this direction and it is a difficult problem because it is a phase separated system. What is the status.

I. Bloch It is a good example about how theory and experiment work together.

We have measured on the superfluid side the T_c, how the T_c goes down. So the phase diagram in the superfluid phase has been measured now.

A. Georges A somehow related question: one of the key issues is whether you will be able to cool fermions cold enough. So what of the promising route to achieve that experimentally?

I. Bloch Let me go back to the shell structure. I think that there are two ideas that are around now: 1. Basically, the first one is related to using tricks of laser cooling; for instance, immerse the system that you want to cool in a BC that will serve as a reservoir and use later laser cooling tricks to basically transfer the entropy of the lattice system to the phonon bath of the BC, to heat he BC. 2. Another idea that people have put forth is that the entropy sits in the shells of the system so if we are talking about an N=1 Mott insulator surrounded by the superfluid shell, basically most of the entropy dominantly sits in the outer shell. This is a good thing of the inhomogeneous system because it separates the region of low entropy and high entropy. The idea would be for instance in an experiment to zap away this very entropic layer, to somehow remove it by some evaporative cooling and then have a very efficient cooling process. This has not been implemented for lattices but Jason Ho did some estimates of this process and it turns out that this should be extremely efficient because all the entropy is really in the shell and you can access those.

F. Wilczek On your summary slide you seem to be hinting that these systems could help with measuring electric dipole moments; what is the idea there?

I. Bloch Basically it is an old idea that was already put forward by Steve Chu: one could use molecules or isolated atoms in these lattices to have very long coherence times; one can think of the lattice as an ideal container because it isolates the atoms from each other. There is no collisional shift or anything; it is like a small and perfect container where the atoms sit and can be interrogated for a long time. The idea of Steve Chu is to do this with Cesium atoms in the lattice and there were ideas to use molecules for that but the general idea is that with one atom or molecule at a site there is no collisional broadening which makes it a very nice environment.

F. Wilczek Do you expect the presence of any kind of collective effect?

I. Bloch No, not at all.

N. Andrei Do you see the decay towards a thermal or non-thermal distribution in the long run?

I. Bloch So far there were only simulations but we are actually working on the experiment and probably we should have some data on that as well. You can see the theory on this slide. When the parameter delta is one, in the isotropic case, you start with a certain magnetization that decays rapidly to zero whereas when you tune to different values you get a different dynamical behaviour of the system. Looking at the yellow points (you can see it already for point five), if this interaction is made weaker you see many many more oscillations and the system is much less damped. If you go to this regime it gets overdamped but

with a much longer time constant that you can see here.

N. Andrei What I have in mind is the experiment of David Weiss where he relaxes the boson gas which is integrable and he sees that it is not thermal. There is an interesting question there because it is in a trap which breaks integrability but you may think that if is weak enough then it is protected by KM-like theorems. Have you any comment about this?

I. Bloch You are completely right; there should be a revival somehow at later times in this. It seems that in the spin model language you can simulate it pretty far. If you take the full Hubbard dynamics it is actually very demanding numerically and people have not been able to go to the point where numerically they would see revivals. Experimentally this is a completely open question; the problem that we have right now, experimentally, is that we have tens of thousands of 1-D systems and all inhomogeneities average out the signal but if you go to single tubes which is a completely different situation and then look at those experiments again. About your point on integrability I think that with the superlattices it would be fantastic to exactly test this: you start out with a single 1-D, perfectly integrable system and then you couple to just another chain and you see how the dynamics changes. So this is the most minimal coupling that you could make: just another single chain, and then look at the effect on integrability. This is indeed a very interesting question.

P. Zoller The next talk will be given by Steven White about Density Matrix Renormalisation Group and related ideas.

Rapporteur Talk by S. R. White: Numerical Methods Using Matrix and Tensor Product Wavefunctions

1. Abstract

Recently, a variety of new simulation methods have been developed based on matrix and tensor product states. We review the definition and motivation for using these states, and give a few examples of some of the most interesting new developments. In particular, we compare the projected entangled pair states method (PEPS) with traditional finite-width DMRG for two dimensional systems.

2. Introduction

Numerical methods for quantum condensed matter systems have continued to increase in both importance and capability. A variety of methods exist, each of which is useful for a particular type of system or for a particular set of properties. For example, density functional theory[1] (DFT) is crucial for gaining an initial understanding of almost any solid, and for detailed properties of systems without strong correlation. This is an example of an "all-electron" method, although this term

is not entirely correct because usually a pseudopotential is used to describe core electrons. For systems with some strong correlation, dynamical mean field theory[2] (DMFT) can be used in conjunction with DFT. Other all-electron methods exist[3,4] which can be useful when one needs greater accuracy (at greater computational cost) than is easily obtained with DFT.

Another class of methods are used to study more approximate model Hamiltonians, with one or two orbitals included per active atom. These methods are most important for systems with very strong correlation. Quantum Monte Carlo (QMC) methods based on loop algorithms are extremely powerful when applied to systems without sign problems.[5–7] They are "exact" (except for statistical errors) and can be used to treat very large two-dimensional and the three-dimensional systems with high enough accuracy to determine critical exponents. Another type of QMC, Green's function Monte Carlo,[8] uses a trial state to reduce or eliminate the sign problem, at the expense of introducing a systematic error which can be hard to determine or control. Cluster versions of DMFT[9] can be used on two and three dimensional fermion systems, and provide detailed thermodynamic and dynamical information, but also have hard-to-determine errors.

Each technique has significant weaknesses. For example, current DFT algorithms lack a proper treatment of correlations between widely separated electrons, e.g. on different atoms. The most powerful methods—in the sense that they can be made exact, at least in principle, by adjusting a knob in the program—tend to have an exponential growth in the computation time as one varies some parameter. For example, the loop algorithm QMC methods can be used for any problem, in principle, but for models with a sign problem, the average sign falls exponentially with the space-time volume of the system, making the statistical error grow exponentially. The density matrix renormalization group[10,11] (DMRG) method—the most important of the matrix product state methods considered here—can, in principle, be used for any system, but in two dimensions the calculation time grows exponentially with the width of the system. This exponential growth is much milder than that of the sign problem in QMC, and the accuracy in 1D is very high, so moderately wide strips can be studied with DMRG, despite the exponential.

The apparent lack of any exponential growth in the computation time for the projected entangled pair states (PEPS) methods for a wide variety of 2D spin models with or without frustration has stimulated great interest.[12,13] PEPS wavefunctions are examples of tensor product states, a natural generalization of the matrix product states underlying DMRG. The lack of an exponential does not mean that these methods are fast. In DMRG, the accuracy is controlled by the size of a matrix, m. The calculation time per site varies as m^3. For the current PEPS algorithms, the calculation time per site grows roughly as D^{10}, where D indicates the size of a tensor index. The key advantage of PEPS is that D varies weakly with the system size, whereas in DMRG m grows exponentially with the system width.

PEPS is one of a variety of new approaches introduced or stimulated by ideas and people from the field of quantum information (QI). DMRG methods arose inde-

pendently from QI, but we now know that QI is the natural language for describing DMRG. Understanding at least some QI ideas is now essential for anyone utilizing DMRG and related methods.

The remainder of this paper will give a brief overview of matrix and tensor product state methods, with a few specific examples of recent developments. First, in Section 2, we will introduce matrix product states, and motivate their usefulness from two different perspectives: 1) as a natural low-entanglement approximation, and 2) as a class of variational states with very nice properties. In Section 3, we will describe a few examples of generalizations of DMRG: a method for periodic boundary conditions; a method for infinite systems; a method for critical systems; and PEPS. In Section 4, we compare the current capabilities of traditional "strip" DMRG and PEPS, and in Section 5 we conclude.

3. Matrix Product States

Let us define a matrix product state (MPS) to describe the wavefunction of a set of N $S = 1/2$'s. It is defined as

$$\psi(s_1, \ldots, s_N) = A_1[s_1]A_2[s_2]\ldots A_N[s_N] \tag{1}$$

Here s_j runs over the states at site j, e.g. state 1 is $\uparrow$ and 2 is $\downarrow$, and $A_j[s_j]$, for a specific value of s_j, is an $m \times m$ matrix. Thus at each site j, two matrices are needed, $A_j[1]$, and $A_j[2]$. The first and last A's are vectors, so that the entire product is a number, which is the value of the wavefunction given $s_1 \ldots s_N$. An alternative form for an MPS is

$$\psi(s_1, \ldots, s_N) = \mathrm{Tr}\{A_1[s_1]\ldots A_N[s_N]\}. \tag{2}$$

In this case all the A's are $m \times m$ matrices. The first form is natural for open boundary condtions; the second, for periodic.

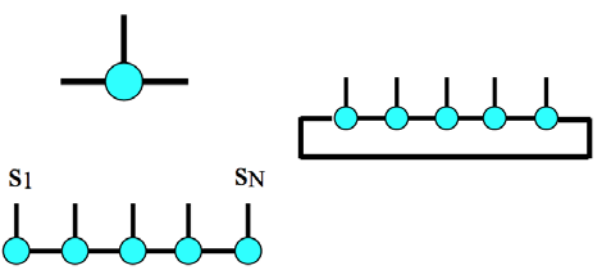

Fig. 1. Diagrams describing matrix product states. The diagram for a single matrix $A[s]$ is shown in the upper left. The vertical link corresponds to the states of a site, while the two horizontal links (with dimension m) connect to adjacent sites. The lower left shows the matrix product used for open boundary conditions. The right figure shows the diagram corresponding to the trace formula, used for periodic boundaries.

It is very useful to describe MPS states and their generalizations using diagrams. Figure 1 shows the diagrams for the two versions of an MPS. The external legs correspond to the values of the spin states, and all internal lines are matrix indices which are summed over. The diagrams generalize in a very natural way to tensor product states.

Matrix product states arise naturally in a variety of contexts. For example, the Affleck Kennedy Lieb Tasaki (AKLT) state for the $S = 1$ chain, which is the exact ground state of a generalization of the Heisenberg model, can be written as a matrix product state composed of 2×2 matrices.[14] Matrix product states describe the wavefunctions appearing in Wilson's numerical renormalization group, although they were not emphasized by Wilson, and were carried over into DMRG, originally developed as a generalization of Wilson's approach.[15] Their usefulness in DMRG was first explained by Östlund and Rommer.[16] Here, we will motivate matrix product states from two other viewpoints: first, as a natural low-entanglement approximation, and second, as a class of variational states with very nice properties.

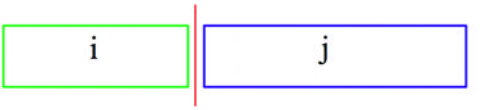

Fig. 2. The partitioning of the system into two parts, the left with states i, and the right with states j.

The entanglement entropy of a wavefunction is defined for a specific partitioning of the system into two parts, and does not involve the Hamiltonian. Let the left part have states i and the right part have states j, so the wavefunction is ψ_{ij}. Regarding the wavefunction as a matrix, we can write

$$\psi_{ij} = \sum_{\alpha} U_{i\alpha} w_{\alpha} V_{\alpha j}. \tag{3}$$

This is nothing more than a singular value decomposition, which can be performed on any matrix. The rectangular matrix U obeys $U^{\dagger}U = 1$, while V obeys $VV^{\dagger} = 1$. The singular values w_{α} are nonnegative real numbers, and if ψ is normalized, $\sum_{\alpha} w_{\alpha}^2 = 1$. If i and j run from 1 to I and J, α runs from 1 to $\min(I, J)$. This matrix factored form is called the Schmidt decomposition in quantum information. We regard w_{α}^2 as the probability of a product state of the system defined by the corresponding column of U and row of V. The entanglement entropy is defined as

$$S = -\sum_{\alpha} w_{\alpha}^2 \ln w_{\alpha}^2. \tag{4}$$

Suppose the entanglement entropy is small. Then the w_{α} must fall off rapidly with α, so it is natural to make an approximation to ψ by truncating the range of

α, say $\alpha = 1 \ldots m$, eliminating the corresponding low-probability Schmidt states defined by the rows and columns of U and V. For a 1D system, the natural partitions are between left and right sides at any link. If one applies this Schmidt decomposition and truncation procedure at every link, one obtains a matrix product state. Thus, an MPS is a natural and compact description of a 1D system with low entanglement. This low entanglement description of an MPS is completely equivalent to the reduced density matrix derivation of DMRG. In addition to the wavefunction, DMRG specifies an efficient method of optimizing the A's, involving minimizing the energy by varying one or two adjacent A's at a time, and sweeping through the sites. The generalizations of DMRG described below also involve similar local optimization and sweeping, but we will omit the specific details.

Because of the log in the definition of the entropy, the number of states kept varies as the exponential of S. We can use known properties of S for various systems to predict the efficiency of an MPS description. For a 1D noncritical system, S is independent of the length of the system, and therefore we expect a fixed m to describe such a system efficiently. For a 1D critical system, it's known that the entropy varies as $\ln(N)$,[17,18] translating to m varying as a power of N, which makes 1D critical systems of hundreds or thousands of sites tractable.

We now consider matrix product states as a class of variational states with very desirable properties. This viewpoint has provided the most general framework for generalizations. Ideally, what properties would one like in a class of variational states? Let us make a list: 1) Exactness: we would like the states to become exact for any state when the number of degrees of freedom is taken to infinity. In the case of MPS, they are exact as $m \to \infty$. 2) Rapid convergence: one would also like rapid convergence as the number of parameters increases. For 1D systems, MPS work well, but for 2D, the convergence is much slower. 3) Physical motivation: consider the diagram for an MPS, and look at two matrices connected by a link. The summation over the link variable corresponds to a summation over fluctuations between the left and right hand sides of the system. A near neighbor link supports the fluctuations important for the terms in a local Hamiltonian, and a small value of m would work well for a single pair of sites. With larger values of m, the extra states can act as a conduit to carry longer-range correlations between longer distance parts of the system. 4) Compactness: we are all familiar from a computer context of compressing or zipping up a file. We would always like to express the wavefunction with as little storage as possible. An MPS replaces an exponentially large wavefunction with a set of $m \times m$ matrices, achieving excellent compactness for 1D systems. 5) Computational convenience: the computation time for DMRG varies as Nm^3, with a very small coefficient provided m is "ramped up" during the sweeps. An analytic example of a compact description without computational convenience is the Bethe Ansatz, for which computing correlation functions is very difficult. Computational convenience is the key issue for many of the generalizations of DMRG–they may have very compact descriptions, but nevertheless can be very slow to compute.

4. Generalizations of DMRG and MPS

Here we consider some illustrative generalizations of MPS or at least the DMRG viewpoint of MPS. The first is for periodic boundary conditions. The original formulation of DMRG was for open boundaries. From the entanglement viewpoint, an open system in 1D has lower entanglement because a partition introduces a single cut, rather than two. By the area law of QI, one expects with an MPS of type Eq. (1), $m_{\text{periodic}} \approx m_{\text{open}}^2$. Östlund and Rommer pointed out the natural version for periodic boundaries, Eq. (2). However, their numerical approach was inefficient, and they only utilized $m = 12$. Verstraete, et. al. reemphasized the naturalness of the trace form, and introduced a much better algorithm for optimizing the A's.[19] They showed that with the trace form, for equivalent accuracy, $m_{\text{periodic}} \approx m_{\text{open}}$. Their computation time varied as Nm^5, slightly better than the traditional DMRG treatment $N(m^2)^3$. More recently, Pippan, et. al. have given an improved algorithm with computation time varying as Nm^3.[20]

The next generalization we will discuss is for infinite translationally invariant systems. A very natural state to propose is the infinite state where the same matrix A is duplicated on every site in the lattice. Unfortunately, with this state it is very hard to optimize the energy efficiently. For computational convenience, it is much better to use as one's ansatz the pattern $\ldots ABABAB \ldots$ defined by two matrices A and B. In this case one can use a Trotter imaginary time evolution to optimize the energy. In this approach, first the odd links have an exponential of the odd-link Hamiltonian applied, and then similarly for the even links, making up one time-step. This forms the infinite time evolving block decimation method (iTEBD) due to Vidal.[21] The Trotter time evolution is also extremely useful for real-time evolution, in order to study dynamical properties of 1D systems.[22–24]

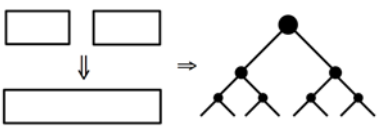

Fig. 3. The left part illustrated the block doubling of a real space RG approach, which corresponds to a tensor product state in the form of a tree diagram (right).

DMRG was originally devised as an effort to fix real space RG approaches for quantum systems. For computational efficiency, the doubling of a block at every step, which is characteristic of most real space approaches, was replaced by adding a single site, which naturally leads to an MPS. For critical systems, the logarithmic growth of the entropy with system size limits the size that can be studied. The MPS also does not exhibit scale invariance in a natural way. A block doubling RG approach would be represented by a tensor product state in the form of a binary tree, as shown in Fig. 3. The tree represents some local entanglement effectively,

but for a partition between two large blocks, say at the midpoint of the tree in Fig. 3, all the entanglement must be transmitted through a single link. Thus the upper links must steadily increase in dimension, and the block doubling approach does not exhibit scale invariance.

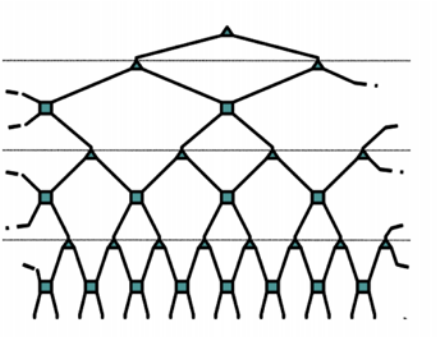

Fig. 4. Tensor product state for MERA.

Recently Vidal has developed a very nice tensor network for 1D critical systems, called the multiscale entanglement renormalization ansatz, or MERA.[25,26] MERA seems to be the right way to perform a quantum real space RG. The tensor product network is shown in the Fig. 4. The bottom external links represent the sites of the chain. The initial two site blocks are centered below the triangles, not the squares. The triangles and squares denote the two types of tensors in the network. The squares are called disentanglers, and they are unitary when one considers the top two links as one index and the bottom two as another. This means that no states are removed as one goes up one layer through the disentanglers. They serve to remove part of the short range entanglement between the ends of adjacent blocks. The three legged triangle tensors, which are row unitary between the top and bottom (referred to as isometries in QI), have fewer degrees of freedom at the top, and perform the truncation of the states in going up to the higher level and length scale. With this approach the entanglement gets organized at different length scales in different layers. Entanglement is transmitted at all layers, so there is no bottleneck between large blocks. The optimization of this state is complicated, but involves minimizing the energy while keeping constraints on the tensors to keep their properties, such as unitarity. For a critical system, one can impose both translational and scale invariance, namely, the tensors are the same in both the horizontal and vertical directions. Very interestingly, for a system described by a conformal field theory, this state directly yields the CFT central charge and scaling dimensions of the primary fields. The method also produces accurate correlations at very large distances, for example at $x \sim 10^9$. The weakness of this approach is computational efficiency: the calculation time grows as m^9. Calculations on the transverse field critical Ising

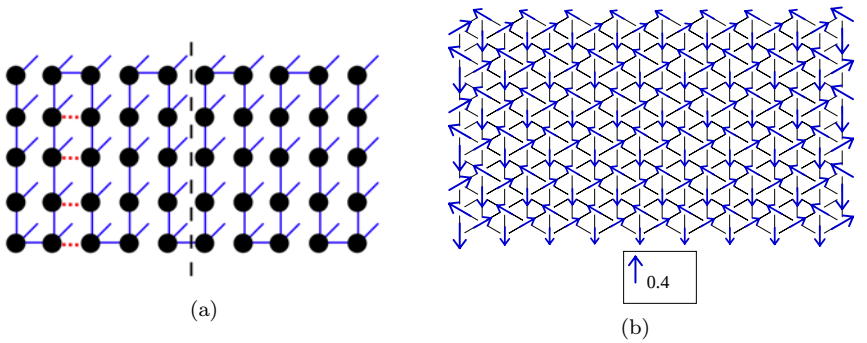

Fig. 5. (a) Matrix product state/DMRG approach for finite width 2D systems. (b) Results for the antiferromagnetic Triangular Heisenberg Model.

model have been performed with up to $m = 6$, for which the error in the energy is about 10^{-7}—which implies very impressive compression of the wavefunction.

5. Two Dimensions

First consider the traditional DMRG method, where the variational state is an MPS. Fig. 5(a) shows the approach for a finite width system. The line connecting all the sites shows the path of the MPS as it winds through the lattice. The dotted lines indicate long-range bonds, which are short range from the point of view of the physical system but long range for the MPS. The vertical dashed line shows a cut in the system partitioning it into two different parts. The area law from QI say that the entropy grows as the width of the system, L_y. That means that the matrix dimension m varies as $\exp(aL_y)$, for some a. The calculation time is proportional to $L_x L_y^2 m^3$. With current workstations, we can use up to $m \sim 5000$, which allows $L_y \sim 8 - 12$ for a Heisenberg model.

As an example of this approach Fig. 5(b) shows results for the triangular lattice Heisenberg model.[27] The triangular lattice Heisenberg model has long-range 120° anti-ferromagnetic order. The simulation above is with cylindrical boundary conditions and with pinning fields on the left and right open edges. In this calculation the error in the energy is around 0.3% and the error in the measurement of a local spin operator is about 0.01. The combination of cylindrical boundary conditions and pinning on the edges allows very efficient extrapolation of the order parameter to the thermodynamic limit.[27] Here, the sublattice magnetization for the infinite system is found to be 0.205(15)), in units where the maximum possible result would be 0.5.

Projected entangled pair states provide a much more natural description for two dimensions.[12,13] PEPS are defined by tensor networks, with the case of the square lattice shown in Fig. 6. The basic unit is a tensor with five indices, four of them directed to the nearest neighbor sites, and the fifth labeling the local degrees of

freedom of a site. We label the maximum value of one of the link indices D. The local correlations between every adjacent pair of sites are directly represented by an index which is summed over, in contrast with the MPS case. For a state with short range correlations, one expects D to be independent of system size, regardless of the number of dimensions. It has also been shown that a finite D PEPS can possess power law correlations,[28] and in test cases very small D give fairly accurate results. The key issue is the computation time for the optimization of the tensors and the contractions in order to calculate observables. Verstraete and Cirac devised a clever approach for these operations without any exponential dependence on the system size, but the computation time is still large $\sim L_x L_y D^{10}$. In addition, the optimization requires an imaginary time evolution requiring thousands of time steps, which is not included in the above asymptotic form. Currently the largest feasible value of D is about $4-5$.

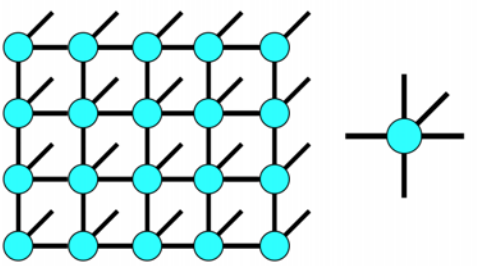

Fig. 6. Tensor network describing a PEPS.

Which method is currently more useful for two dimensions, PEPS or traditional strip DMRG? Recently some PEPS results for $D = 4$ have been published for the square lattice Heisenberg model, for open systems of size 8×8 and 10×10.[29] A comparison of these results with those of traditional DMRG are shown in Table 1.

Table 1. Comparison of DMRG and PEPS for open square lattice Heisenberg systems.

System	PEPS D=4	DMRG m=200	DMRG m=400	DMRG m=1600	DMRG m=3600
8×8	-39.414	-39.400	-39.539	-39.615	
10×10	-62.493	-62.155	-62.490	-62.777	-62.846
12×12		-89.783	-90.501	-91.175	-91.325

One way to evaluate the results is to ask: what value of m does a given value

of D correspond to? Since the states are variational, we take the energy as the appropriate benchmark. In the 8×8 case, $D = 4$ PEPS corresponds to $m \approx 200$, and the total energy is off by about 0.5%, using larger m DMRG to estimate the exact result. For the 10×10 case, $D = 4$ corresponds to $m \approx 400$, and the energy is off by about 0.6%, hardly different from the 8×8 case. For the 12×12 case, the DMRG results at $m = 3600$ are accurate to about 0.1%. We estimate they would be off by about 0.6%, and thus correspond to $D = 4$ PEPS, at about $m = 800$. In terms of compression, PEPS is much more efficient, with a storage of 4^4 per site, compared to 400^2 for DMRG for the 10×10 case. On these system sizes DMRG can reach much higher accuracy, because of its superior computational efficiency. On larger sizes, say 20×20, PEPS would be more efficient. A key issue then becomes the extrapolation to the thermodynamic limit: how efficiently can that be done with systems of order 20×10, which are accessible with a traditional DMRG treatment with very good accuracy.[27]

There have been a number of improvements to PEPS proposed. One improvement is to treat an infinite system using the analogous treatment for 1D mentioned earlier, with an $\ldots ABAB \ldots$ pattern.[30] So far the largest D value reported for this iPEPS approach is $D = 3$. Another approach is to combine PEPS with quantum Monte Carlo methods,[31] although the initial work was in 1D.

6. Conclusions

A variety of computational methods continue to grow in importance for the modern study and understanding of condensed matter systems. In the case of matrix product state/DMRG approaches, the field of quantum information has helped provide a new set of tools, leading to a variety of generalizations and improvements of these algorithms. Especially notable are tensor network approaches, such as PEPS. Among the challenges are optimizing both the compactness of a representation of the state and its computational efficiency. We have only touched on a few techniques here. Among the most notable things left out are real-time evolution, finite temperature methods, techniques for disordered systems, and for classical critical and nonequilibrium systems.

Acknowledgments

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Discussion

P. Zoller We have time for discussion.

M. Freedman I would like to make three brief mathematical comments on your very interesting talk. One concerns the passage to be done from the trees and the networks including disentanglers. It seems that it could be a step like using efficient communication graphs. Apparently the problem with the tree was the bottleneck of the top node and of course there is a whole theory in graph theory for producing efficient communication.

The second point concerns the use of the singular value decomposition to truncate your matrices. That represents a decision in non-linear signal processing. Of course people for a long time have worried which basis to use to keep the most important information. There are a lot of choices there – the single value

is just one – for different applications you can get more information about the ground state wave function by using, whatever, wavelets or something else.
The third comment is: the whole subject basically seems to call out coarse graining, that is, you have this very complicated tensor network and you want to approximate its partition function with a network that is computationally manageable. I think that there are other areas in physics where one can learn how this coarse graining might go. The whole subject goes back to the late sixties with Roger Penrose and his tensor network that was also a genesis for loop gravity I think. I just wonder whether the dread high energy experts might know some ways of dealing with these tensor networks.

S. White Let me just comment on that last part. I learned a lot from talking to the people in this conference. I learned from Matthias Troyer that one of his students had found some tensor network in quantum loop gravity and anyway it goes back a while there. I did not talk a lot about finite temperature but it seems that there is a technique there like purification that has been reinvented again and again. No one knows all of physics so these things happen quite a bit, but it is wonderful that different domains merge together and that we can learn so much from quantum information. I hope that some of it will go the other way too.

S. Das Sarma Steve, you described the state of the art in quantum numerical techniques but for me when I do large numerical work, I am usually interested in a very specific physical question. For example as you know we have been trying to understand whether the 5/2 quantum Hall state, the actual state is spin polarized or not and (?) has been doing this heroic DMRG calculation. It required to run a computer cluster for a year. Other example, I may be interested in knowing whether there could be a non-Abelian state at $\nu = 1/2$ for some quantum Hall systems for some thickness and I have been looking at it with colleagues at Maryland and again it is a very hard problem. To conclude, can you give us some examples of actual physical questions where the techniques described by you have led to concrete and definitive results?

S. White Well, maybe the best example is the results that Ian Affleck showed earlier today in a one dimensional system.

S. Das Sarma One dimension I know, could you tell us something about higher dimensional systems?

S. White OK, for 2 dimensions we have been working our way up with ladders and for narrow ladders the results are quite good, as precise as we would like. If you go to larger ones it is less clear but I think that we are getting some real conclusions on that.

S. Das Sarma Is there any hope for extending these techniques to the 2-D Hubbard model?

S. White Well we may find that PEPS is getting good enough to do that some time soon. Right now for the 2-D Hubbard model we can do a width 6 system and we get reasonable results there. We do find a tendency for striping there.

Again we can go in a certain width. We also have other numerical methods such as the cluster DMFT and also the versions of quantum Monte Carlo that have a trial state to control the sign problem. Those results all give answers, sometimes it is difficult to know what the errors are but we can put a lot of pieces together.

X-G. Wen I would like to make a comment: these matrix product states and tensor product states not only have the advantage of providing very efficient numerical simulations but also present very important conceptual advantages, I mean, thinking about this topological phase, the essence of topological phase is the long range entanglement, and these tensor product states actually are concrete representations of long range entanglement in the sense that one can prove that all the non chiral topological order can be represented as a tensor product state.

So therefore that is a way to answer to Shankar's question that if you use these tensor product states as a trial wave function you can in principle calculate the phase diagram which includes both the symmetry breaking states and the topologically ordered states.

S. Sachdev I would like to elucidate a very elementary confusion. I do not understand why it is not always better to work on translationally invariant systems. If I understand correctly one just has a wave function with M squared variational parameters so why is it not simpler to work with them?

S. White There is an interesting paper by Moore. I do not know whether this is a pertinent comment but if you use these translationally invariant systems in a critical system you cannot use M=infinity so you use a finite M. That M translates into a correlation length and so the question is: "What should you do ?" In DRMG what I would do is take a system of a size hundred sites, two hundred and then go up to a system of a few thousands and then I would ask to Ian Affleck how that scaled and we would fit and do the finite size scaling. The other way is to work directly in this infinite system and then you increase M and see how the result converges, but it translates into a finite correlation length and so it is similar to a finite system.

S. Sachdev So you think that in a finite system you keep M smaller but yet build up correlations better? Is that what you mean?

S. White Yes. So you can think of the lattice size as a regularizer that we can just nail it and get exact diagonalization results, up to a certain size.

W. Ketterle – prepared comment About cold fermions.

C. Varma With these polarized fermions could you observe experimentally phase separation of the polarized part?

W. Ketterle Yes exactly. In this phase diagram that shows polarization and temperature we have here the balanced gas and if we polarize it more and more we quench the superfluid and here we reach the Pauli or Clogston-Chandrasekhar limit of superfluidity and this diagram is superfluid but here is a first order phase transition and here is a second order phase transition. So in this region

we have a phase separation between the excess fermions and the superfluid system.

C. Varma How do you see experimentally, with this technique that there is a phase separation?

W. Ketterle We have a direct imaging technique to see the spin up and the spin down fermions and we see a region where both spins are matched and then there is another region where the excess fermions reside.

C. Varma And this happens completely without too many interfaces?

W. Ketterle The interface that we observe is pretty sharp. We use tomographic techniques – since we have a 3 dimensional cloud we have a line of sight integrated optical imaging technique and we have to use tomographic reconstruction.

C. Varma But you have one interface and not several interfaces?

W. Ketterle We see one sharp interface so we see a superfluid bubble of matched populations surrounded by a shell which we think is normal where the population is imbalanced. And we see that the sharp interface disappears when we raise the temperature and that is how we infer the tri-critical point.

M. Cohen I wonder whether you could comment on the nature of the pairs. Obviously as you tune the Feshbach resonances you can make the interaction stronger and go over to something that is more like Heitler-London pairs which would satisfy Bose statistics and Bose commutation relations. However for a true Cooper pair the interaction can be arbitrarily weak, and it is just the presence of the Fermi surface that allows you to bind the particles for arbitrarily weak interactions . The resulting Cooper pairs do not satisfy Bose commutation relations, hence they are not bosons. So is there a place where you see a difference between true Cooper pairs and Heitler-London pairs somewhere in the cross-over region or do you distinguish these at all?

W. Ketterle Well, it is the nature of the cross-over region that there is no sharp transition and, to be truthful I must say that we are not able with those Feshbach resonances to scan from the extreme BEC limit to the extreme BCS limit. What happens is that if you go too far to the BEC side the molecules do bad collisions, so we have mainly explored the region where $kF \cdot a$ (kF is the fermion momentum and a the scattering length) varies from plus one to minus one. So you would not expect a very strong change in the nature but we did characterize the pairs: we measured the pair size through spectroscopy and we saw that there were changes that the cross-over theory predicted. But I should also say that at a $kF \cdot a$ of minus one the pair size is still comparable to the interatomic spacing so we have not reached the limit where the Cooper pairs, and the Cooper pairs that you envision – namely pairs for which the correlation length, the coherence length is much larger than the interparticle spacing. On the other hand we find that it has been so beautifully studied in metals that it would even not be a major goal for us to go there because the temperatures become exponentially small and are currently not in reach for us.

S. Das Sarma – prepared comment On quantum Hall in the context of cold atoms.

P. Zoller Are you concerned that if one takes this standard set of simply atoms in an optical lattice and deriving some effective Hamiltonian in some higher order perturbation theory that the overall energy scales might become very small and that therefore the corresponding times scale and temperature requirements and so on.

S. Das Sarma Absolutely. I am very much concerned about it and I think that for each of the proposed effective Hamiltonians the people who are proposing it actually have a responsibility of estimating the energy and time scales in order to guarantee the physical realisability of the experiment, so, yes I am very much concerned by this question.

N. Read I want to go back to your remarks about rotating atoms, bosons and quantum Hall type states. We actually did some numeric work on this, and the cases where we have evidence that there is a non-Abelian state are as follows: we have strong evidence for filling factor 1, we have fairly good evidence for 3/2, and we have from the original work of Nigel Cooper and collaborators some weaker evidence I would say for fillings larger, k/2 for larger values of k=4, 5, 6, ..., up to filling factor around 6. There may be some recent work on a few other fractions, but for arbitrary fractions we just do not know the physics because they have not been and cannot be studied convincingly, on account of the usual finite size limitations. The same problem exists for fermions, of course, but we have numerical evidence for non-Abelian states at fractions 5/2, 7/2, 12/5, 13/5, and so it's hard to make a case that there are more or fewer non-Abelian quantum Hall states for bosons than for fermions.

S. Das Sarma For 12/5 or 13/5 I simply do not agree that they are definitively established to be non-Abelian states but this is another discussion. I agree on the comment that you made.

N. Read I didn't say those are "definitively established", but I believe the evidence is stronger than for say bosons at filling factor 5/2.

M. Freedman Just with regard to rotating bosons for creating quantum Hall states there is an interesting proposal of Anderson and Slingerland, at filling factor equal $\nu = 6$, to get the Fibonacci anyons. In that theory one finds the Lie group G2 at level 1, and this is a very interesting non-Abelian effect. What makes it particularly interesting is that since the filling fraction is large, 6, it has to rotate slowest of all this cases.

B. Altshuler – prepared comment On Mott insulators and Anderson localization.

A. Georges I am a bit confused by the comment on the interactions actually. The fact that we have a lattice, if we have no interactions, makes it a perfectly appropriate setting to understand Anderson localization, right ? So I guess that your comment on the Firenze experiment is about interactions but I imagine that those could be tuned to zero?

B. Altshuler I am afraid that I probably confused you. What I wanted to tell is that you can prepare systems already in the Mott insulator phase. So if you start with a Mott insulator and add some disorder then you only have a gap in the spectrum and what you should expect is that disorder will create some tail in density of states and in the beginning maybe there will still be a gap although it is strongly depends on the particular model. So what I wanted to tell is that depending on how close you are to the Mott insulator in the clean case the character of the transition to real insulator in the presence of disorder can change.

A. Georges That is not what is done in Inguscio's experiment. Is it?

B. Altshuler I cannot comment; I have not studied Inguscio's experiment well enough to go into details. I think that he started with superfluid states and that there was a transition to insulator so it is not probably this situation but opposite. However people who discussed it, I do not remember from which group, discussed these two steps and even talked about two transitions.

T. Arecchi My remark is just very short. In the case of Inguscio's experiment the scattering length was 0.15 Bohr radius which means that the atoms were practically not interacting even though the density was higher than in the previous experiments.

G. Parisi Let me add a comment on Inguscio's experiment. The experiment was done on a quasi-periodic lattice and not this kind of non-regular lattice.

T. Arecchi I think that both experiments were done, on a quasi-periodic lattice and also with speckle. There were several papers like five of them on the subject.

G. Parisi It was quasi-periodic and weakly interacting so it was more like Anderson.

A. Georges – prepared comment On cold atoms in optical lattices.

M. Troyer – prepared comment On quantum Monto Carlo and comparison with cold atom experiments.

I. Cirac – prepared comment On PEPS.

D. Haldane Can you describe a free electron metal with a Fermi surface with a PEPS tensor?

I. Cirac Yes, in principle you can do that, because we know that, we have not done it, but in one dimension for instance...

D. Haldane No, not in one dimension sorry, in higher dimension.

I. Cirac Well, in higher dimensions, as long as it would fulfill the area law up to logarithmic corrections, then I believe that it would be possible, I mean we have not tried to do that.

D. Haldane That is something that you could easily test against an exact result.

I. Cirac Yes that is true.

G. Parisi Can you make a comparison of the advantages or disadvantages of this method with respect to Monte Carlo.

I. Cirac Well, if there is no sign problem which means that you can use Monte Carlo, then use it – I do not talk about millions of spins, I say 400 spins at most

– but in the case that there is a sign problem you can go to this lattice which is reasonable; with that and with DMRG also that was mentioned in the talks, there are other options to Monte Carlo.

Prepared Comment by Assa Auerbach[a]: Transport Studies of Lattice Bosons: Paradigms for Fluctuating Superconductivity

1. Strongly Fluctuating Superconductors

"Conventional" superconductors undergo a pairing transition at T_c, which can be well described by BCS mean field theory.[1] In general they have large superfluid density n_s, (e.g. in two dimensions $\hbar^2 n_s/m >> T_c$), and weak phase fluctuations.

In contrast, under-doped high T_c cuprates,[2,3] small capacitance Josephson arrays, and disordered thin films,[4] are characterized by low superfluid density. This enhances the role of phase fluctuations and vortex delocalization near T_c. Pairing correlations persist well above T_c,[5] and "normal state" transport coefficients[6] do not follow familiar Fermi liquid behavior.

To describe strongly fluctuating superconductors, it is natural to consider effective Hamiltonians of charge $2e$ bosons.[7,8] Continuum and weakly interacting superfluids are well approximated by the Gross-Pitaevski (GP) equation.[9] However, a strong periodic potential generally enhance the short wavelength fluctuations beyond the validity of the GP approach. Non uniform potentials are unavoidable in solid state superconductors. Nowadays, their effects can be systematically studied in cold atom condensates on optical lattices.[10] At strong interactions and commensurate fillings, the superfluid is unstable toward charge-gapped Mott insulator phases, or "vortex condensates".[11] The phase diagrams of lattice bosons have been studied extensively in recent years. However, little is known about their vortex dynamics and transport coefficients, especially in the strongly interacting regime.

Here we report some recent results[12] on hard core bosons on finite toroidal clusters. We find several interesting effects of the lattice on the ground state and Hall conductivity, which may have experimental implications.

1.1. *The Model*

The gauged quantum XY model on a square lattice represents two dimensional hard core bosons in a perpendicular magnetic field:

$$\mathcal{H} = -t \sum_{\langle ij \rangle} \left(e^{iqA_{ij}} S_i^+ S_j^- + \text{H.c} \right) - 2 \sum_i \mu_i S_i^z. \tag{1}$$

The local density fluctuations are given by $n_i = S_i^z + \frac{1}{2}$, and the superfluid order parameter is the magnetization in the xy plane. The mean field superfluid transition

[a]This write-up was coauthored by Netanel Lindner

temperature goes as $T_c = tn_b(1-n_b)$, where $n_b = N_b/N$ is the filling fraction. An important distinction between hard core lattice bosons and continuum bosons, is the existence of a charge conjugation symmetry $C \equiv \exp(i\pi \sum_{\mathbf{r}} S^x_{\mathbf{r}})$. C transforms "particles" into "holes", i.e. $n_i \to (1-n_i)$, and the Hamiltonian into

$$C^\dagger \mathcal{H}[\mathbf{A}, n_b] C = \mathcal{H}[-\mathbf{A}, 1-n_b], \tag{2}$$

where $n_b = N_b/N$ is the filling fraction.

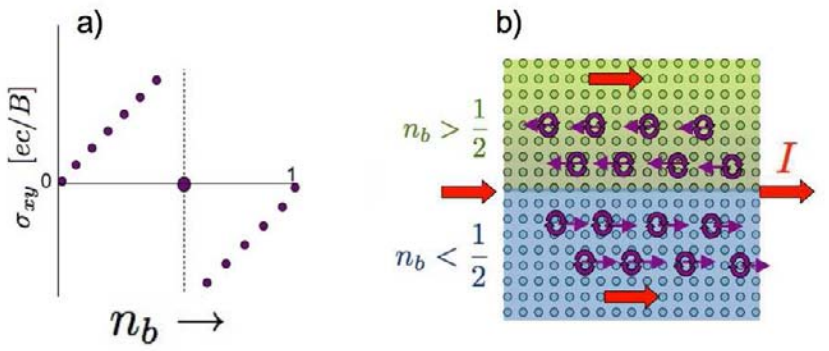

Fig. 1. Reversal of Hall conductivity and Magnus action of hard core bosons at half filling. (a) The zero temperature Hall conductivity given by the ground state Chern number of a 16 site square lattice on a torus. (b) Vortex drift directions (purple arrows) in the presence of a bias current (red arrows), for regions of lower (blue) and higher (green) boson density than half filling.

1.2. *Hall conductivity*

A consequence of (2) is that the Hall conductivity is *antisymmetric* in $n_b - 1/2$:

$$\sigma_H(n_b, T) = -\sigma_H(1-n_b, T). \tag{3}$$

The temperature-dependent Hall conductance of the finite cluster is given by the thermally averaged Chern numbers.[13] A zero temperature Hall conductance as a function of filling for $N_\phi = 1$ is plotted in Fig. 1a. At zero temperature, $\sigma_H = N_b$ below half filling follows the Galilean invariant result $\sigma_{\rm H} \propto N_b/N_{\rm v}$. At half filling, $\sigma_{\rm H}$ reverses sign as expected by (3).

Fig. 1a shows a dramatic effect of the lattice on the Hall coefficient: $\sigma_{\rm H}$ undergoes a sharp transition between $\sigma_{\rm H} > 0$ ($\sigma_{\rm H} < 0$) just below (above) half filling. For hard core bosons $\sigma_{\rm H}(T, n_b)$ decreases with temperature, with a characteristic temperature scale which vanishes at half filling.[12]

In terms of vortex dynamics, Hall conductivity inversion implies that vortices suddenly drift in opposite directions as density of bosons is varied near half filling (see Fig. 1b). We propose to try to observe such a dramatic effect for bosonic atoms

on rotating optical lattices.[10] As the density changes in space with the trapping potential, vortices on either side of the half filling separatrix are expected to flow in opposite directions relative the local superflow.

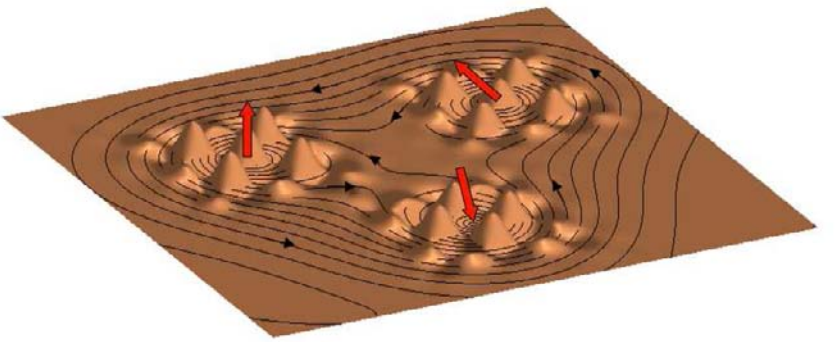

Fig. 2. Illustration of three vortices of hard core bosons at half filling with charge density modulations in their cores. Arrows depict directions of their v-spins.

1.3. *V-spins of vortices at half filling*

At half filling, vortices see no Magnus field, but instead they acquire spin half quantum numbers we denote by 'v-*spins*'. When the gauge field in Eq. (1) describes N_ϕ flux quanta uniformly penetrating the torus, N_ϕ vortices are inserted into the ground state. At half filling $n_b = 1/2$, for any odd number of vortices $N_\phi = 2m+1, m = 0, 1, 2, \ldots$, all eigenstates are at least two-fold degenerate. We have proven[12] that these doublets are associated with SU(2) algebra of local symmetry operators. The v-spin in the "z" direction measures a bipartite charge density wave in the vortex core, as depicted in Fig 2. V-spin interactions between vortices decay exponentially. V-spin excitations are expected to dominate the low temperature thermodynamics at low values of external magnetic field.

1.4. *Vortex mass, and vortex lattice melting*

At half filling, a vortex hops on the dual lattice with half a flux quantum per plaquette. Its hopping rate $t_{\rm v}$ was fit to exact numerical eigenenergies of $\mathcal{H}$. Our results for $N = 20$, show[12] that at half filling, vortices are as 'light' as bosons, $t_{\rm v} \approx t$.

When multiple vortices are introduced by a magnetic field or rotation, they tend to localize in an Abrikosov lattice which coexists with superfluidity. In two dimensions the vortex lattice can melt by quantum fluctuations resulting in a non-superfluid Quantum Vortex Liquid (QVL). A system of interacting vortices can be

mapped to the Boson Coloumb Liquid studied by Magro and Ceperly (MC)[14] . Using our values of $t_{\rm v}$, the critical vortex melting density was bounded by a surprisingly low vortex density,

$$n_{\rm v}^{\rm cr} \leq \left(6.5 - 7.9\frac{V}{t}\right) \times 10^{-3} \text{ vortices per site.} \tag{4}$$

This implies that a QVL is achievable at manageable rotation frequencies for cold atoms on optical lattices, and by moderate magnetic fields for Josephson junction arrays and cuprate superconductors.

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Discussion

P. Zoller Thank you very much to everybody who talked in this session and then I would like to hand the microphone over to B. Halperin who will close the conference.

Closing Session

Address by the Chair of the Conference Bertrand Halperin

We have now come to the end of the 24th Solvay Conference on Physics. From my point of view this has been an extraordinary meeting, one which has far exceeded my expectations. For this success, we owe thanks to a large number of people, and I would like to express here my personal appreciation for their many contributions.

First, I want to thank the rapporteurs who did a fantastic job. It is a very difficult task to summarize a large field, presenting a balanced overview while at the same time giving an insightful perspective. In my view, the rapporteurs have met this challenge with truly remarkable success.

Next I want to thank the session chairmen, who also had a formidable task, to guide the discussions and to keep people on track. I think that the session chairs succeeded in this, and that they did a great job encouraging people to speak up and ask questions

As I said in my opening remarks, I consider the open discussions to be one of the most important features of a Solvay conference. This was reflected in the fact that discussion time amounted to more than half of the conference program. So, of course I also would like to thank all the participants who contributed to the discussions. Your questions were extremely helpful in clarifying points of the presentations, and your spontaneous comments contained many invaluable insights.

I do not forget to thank also those who contributed short prepared comments. I thank particularly those who made an effort to keep their time under five minutes and those who did adhere to the draconian rule of at most two transparencies. I believe that these restrictions really did help to keep things flowing, and I hope that a majority of the participants would endorse that view.

We all owe a special debt of gratitude to the people who worked behind the scenes to make all of this possible. First we must thank the members of the Solvay

Institutes, particularly Marc Henneaux and Alexander Sevrin, at the scientific level, as well as the Scientific Committee and David Gross, who ultimately convinced the Institutes to have this program on condensed matter physics. I must thank Marc and David especially for the very helpful advice they gave me on many points during the organization of the conference. I also received valuable informal advice about the program from a number of other colleagues, but perhaps I should not mention them by name lest they be blamed for any of my mistakes.

I must also thank the Solvay staff, in particular Isabelle Juif and Dominique Bogaerts, who put in an enormous amount of work, for many months leading up to the conference. It will surely be an enormous relief for them when the last taxi has left, but their work will not be completely finished – there will still be the conference proceedings to be produced.

I would like to thank the young people who have been sitting in the background of the conference, taking notes on the discussions. They will be working hard, in the coming months, to make these into intelligible transcripts for publication in the conference proceedings, as records of the discussions are an important part of the Solvay tradition.

The staff of the hotel Métropole deserves credit for its excellent work. Along with the Solvay people, they are responsible for the fact that this has been a most luxurious and well-run conference; just about everything has functioned perfectly.

As I have done before, I would like to thank the Solvay family for their support for the whole conference project, which has now been running for almost a century. I appreciate particularly their efforts in reviving the Solvay Conferences during the last few years, and I wish to thank them most sincerely for their very warm hospitality in welcoming the participants at the conference and at their home.

Finally, I would like to express my gratitude to His Royal Highness Prince Philippe of Belgium for gracing us with his presence at a special session, where we had the opportunity to explain something about the purpose of our conference. His attendance and his interest in our work will stand out, for all of us, as a highlight in our recollections of this meeting.